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Smoothness of functions global and along curves over ultra-metric fields.

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Abstract

The article is devoted to the investigation of smoothness of functions $f(x_1,...,x_m)$ of variables $x_1,...,x_m$ in infinite fields with non trivial non archimedean valuations, where $m \ge 2$. Theorems about classes of smoothness C^n or C^n_b of functions with continuous or bounded uniformly continuous on bounded domains partial difference quotients up to the order n are investigated. It is proved that from $f \circ u \in C^n(\mathbf{K},\mathbf{K}^l)$ or $f \circ u \in C^n_b(\mathbf{K},\mathbf{K}^l)$ for each C^∞ or C^∞_b curve $u : \mathbf{K} \to \mathbf{K}^m$ it follows, that $f \in C^n(\mathbf{K}^m,\mathbf{K}^l)$ or $f \in C^n_b(\mathbf{K}^m,\mathbf{K}^l)$. Moreover, classes of smoothness $C^{n,r}$ and $C^{n,r}_b$ and more general in the sense of Lipschitz for partial difference quotients are considered and theorems for them are proved.

1 Introduction

Fields with non archimedean valuations such as the field of p-adic numbers were first introduced by K. Hensel [7]. Then it was proved by A. Ostrowski [16] that on the field of rational numbers each multiplicative norm is either the usual norm as in $\mathbf{R}$ or is equivalent to a non archimedean norm $|x| = p^{-k}$, where $x = np^k/m \in \mathbf{Q}$, $n, m, k \in \mathbf{Z}$, $p \ge 2$ is a prime number, n and m and p are mutually pairwise prime numbers. It is well known, that each locally compact infinite field with a non trivial non archimedean valuation is either a finite algebraic extension of the field of p-adic numbers or is isomorphic to the field $\mathbf{F}_{p^k}(\theta)$ of power series of the variable θ with expansion coefficients

in the finite field $\mathbf{F}_{p^k}$ of p^k elements, where $p \geq 2$ is a prime number, $k \in \mathbf{N}$ is a natural number [18, 22]. Non locally compact fields are also wide spread [4, 18, 19].

Last years non archimedean analysis [18, 19, 20] and mathematical physics [8, 9, 10, 21] are being fastly delevoped. But many questions and problems remain open. In the classical case it is known the Boman's theorem relating smoothness of a function of several real variables and of its compositions with smooth curves [2, 11]. But this problem was not studied completely in the non archimedean case besides particular cases and instead of curves for compositions with functions of more than one variable [1], that is the significant simplification of the problem.

In the non archimedean analysis classes of smoothness are defined in another fashion as in the classical case over $\mathbf{R}$, since locally constant functions on fields $\mathbf{K}$ with non archimedean valuations are infinite differentiable and there exist non trivial non locally constant functions infinite differentiable with identically zero derivatives [19, 20]. This is caused by the stronger ultra-metric inequality $|x+y| \leq \max(|x|,|y|)$ in comparison with the usual triangle inequality, where $|x|$ is a multiplicative norm in $\mathbf{K}$ [18]. In papers [1, 12, 13, 14] there were considered classes of smoothness C^n for functions of several variables in non archimedean fields or in topological vector spaces over such fields.

This paper is devoted to the investigation of smoothness of functions $f(x_1,...,x_m)$ of variables $x_1,...,x_m$ in infinite fields with non trivial non archimedean valuations, where $m \geq 2$. In the paper fields locally compact and as well as non locally compact are considered. Theorems about classes of smoothness C^n or C^n_b of functions with continuous or uniformly continuous on bounded domains partial difference quotients up to the order n are investigated. It is proved that from $f \circ u \in C^n(\mathbf{K},\mathbf{K})$ or $f \circ u \in C^n_b(\mathbf{K},\mathbf{K})$ for each C^∞ or C^∞_b curve u it follows, that $f \in C^n(\mathbf{K}^m,\mathbf{K})$ or $f \in C^n_b(\mathbf{K}^m,\mathbf{K})$. Moreover, classes of smoothness $C^{n,r}$ and $C^{n,r}_b$ and more general in the sense of Lipschitz for partial difference quotients are considered and theorems for them are proved.

Many specific features of the non archimedean case in comparison with the classical one are found. In the non archimedean case analogs of classical theorems over $\mathbf{R}$ such as 3 and 10 [2] are not true due to the ultrametric inequality for the non archimedean norm, and since if a function f is homogeneous, then $\bar{\Phi}^k$ need not be homogeneous for $k \geq 1$. Theorem 2 from [2]

in the non archimedean case is true in the stronger form due to the ultrametric inequality (see Theorem 39 below). The notion of quasi analyticity used in the classical case in [2] has not sense in the non archimedean case because of the necessity to operate with the partial difference quotients $\bar{\Phi}^k f$ instead of derivatives $D^k f$. It leads naturally to the local analyticity in the non archimedean case. In the latter case the exponential function has finite radius of convergence on $\mathbf{K}$ with $char(\mathbf{K}) = 0$. Moreover, in the proof of Theorem 42 it was used specific feature of the non archimedean analysis of analytic functions for which an analog of the Louiville theorem is not true (see also [19]).

Several lemmas of the paper serve for subsequent proofs of theorems. It is proved in theorems 38-42 below, that for corresponding smoothness, for example, $C^n_\phi(\mathbf{K}^m, \mathbf{K})$ of a function f it is sufficient that $f \circ u \in C^n_\phi(\mathbf{K}, \mathbf{K})$ for each curve $u \in C^\infty(\mathbf{K}, \mathbf{K}^m)$, but a local analyticity of u instead of C^∞ is insufficient.

2 Smoothness of functions

1. Definitions. Let $\mathbf{K}$ be an infinite field with a non trivial non archimedean valuation, let also X and Y be topological vector spaces over $\mathbf{K}$ and U be an open subset in X. For a function $f : U \to Y$ consider the associated function

$f^{[1]}(x, v, t) := [f(x + tv) - f(x)]/t$

on a set $U^{[1]}$ at first for $t \ne 0$ such that $U^{[1]} := \{(x, v, t) \in X^2 \times \mathbf{K}, x \in U, x + tv \in U\}$. If f is continuous on U and $f^{[1]}$ has a continuous extension on $U^{[1]}$, then we say, that f is continuously differentiable or belongs to the class C^1. The $\mathbf{K}$-linear space of all such continuously differentiable functions f on U is denoted $C^{[1]}(U, Y)$. By induction we define functions $f^{[n+1]} := (f^{[n]})^{[1]}$ and spaces $C^{[n+1]}(U, Y)$ for $n = 1, 2, 3, ...$, where $f^{[0]} := f$, $f^{[n+1]} \in C^{[n+1]}(U, Y)$ has as the domain $U^{[n+1]} := (U^{[n]})^{[1]}$.

The differential $df(x) : X \to Y$ is defined as $df(x)v := f^{[1]}(x, v, 0)$.

Define also partial difference quotient operators Φ^n by variables corresponding to x only such that

$\Phi^1 f(x; v; t) = f^{[1]}(x, v, t)$

at first for $t \ne 0$ and if $\Phi^1 f$ is continuous for $t \ne 0$ and has a continuous extension on $U^{[1]} =: U^{(1)}$, then we denote it by $\bar{\Phi}^1 f(x; v; t)$. Define by induction

$\Phi^{n+1} f(x; v_1, ..., v_{n+1}; t_1, ..., t_{n+1}) := \Phi^1(\Phi^n f(x; v_1, ..., v_n; t_1, ..., t_n))(x; v_{n+1}; t_{n+1})$ at first for $t_1 \ne 0, ..., t_{n+1} \ne 0$ on $U^{(n+1)} := \{(x; v_1, ..., v_{n+1}; t_1, ..., t_{n+1}) : x \in U; v_1, ..., v_{n+1} \in X; t_1, ..., t_{n+1} \in \mathbf{K}; x + v_1 t_1 \in U, ..., x + v_1 t_1 + ... + v_{n+1} t_{n+1} \in U\}$. If f is continuous on U and partial difference quotients $\Phi^1 f$,...,$\Phi^{n+1} f$ has continuous extensions denoted by $\bar{\Phi}^1 f$,..., $\bar{\Phi}^{n+1} f$ on $U^{(1)}$,...,$U^{(n+1)}$ respectively, then we say that f is of class of smoothness C^{n+1}. The $\mathbf{K}$ linear space of all C^{n+1} functions on U is denoted by $C^{n+1}(U, Y)$, where $\Phi^0 f := f$, $C^0(U, Y)$ is the space of all continuous functions $f : U \to Y$. Then the differential is given by the equation $d^n f(x).(v_1, ..., v_n) := n! \bar{\Phi}^n f(x; v_1, ..., v_n; 0, ..., 0)$, where $n \ge 1$, also denote $D^n f = d^n f$. Shortly we shall write the argument of $f^{[n]}$ as $x^{[n]} \in U^{[n]}$ and of $\bar{\Phi}^n f$ as $x^{(n)} \in U^{(n)}$, where $x^{[0]} = x^{(0)} = x$, $x^{[1]} = x^{(1)} = (x, v, t)$, $v^{[0]} = v^{(0)} = v$, $t_1 = t$, $x^{[k]} = (x^{[k-1]}, v^{[k-1]}, t_k)$ for each $k \ge 1$, $x^{(k)} := (x; v_1, ..., v_k; t_1, ..., t_k)$.

Subspaces of C^n or $C^{[n]}$ of all bounded uniformly continuous functions together with $\bar{\Phi}^k f$ or $\Upsilon^k f$ on bounded open subsets of U and $U^{(k)}$ or $U^{[k]}$ for $k = 1, ..., n$ denote by $C^n_b(U, Y)$ or $C^{[n]}_b(U, Y)$ respectively.

Consider partial difference quotients of products and compositions of functions and relations between partial difference quotients and differentiability of both types. Denote by $L(X, Y)$ the space of all continuous $\mathbf{K}$-linear mappings $A : X \to Y$. By $L_n(X^{\otimes n}, Y)$ denote the space of all continuous $\mathbf{K}$ n-linear mappings $A : X^{\otimes n} \to Y$, particularly, $L(X, Y) = L_1(X^{\otimes 1}, Y)$. If X and Y are normed spaces, then $L_n(X^{\otimes n}, Y)$ is supplied with the operator norm: $\|A\| := \sup_{h_1 \ne 0, ..., h_n \ne 0; h_1, ..., h_n \in X} \|A.(h_1, ..., h_n)\|_Y / (\|h_1\|_X ... \|h_n\|_X)$.

2. Lemma. *The spaces $C^{[1]}(U, Y)$ and $C^1(U, Y)$ are linearly topologically isomorphic. If $f \in C^n(U, Y)$, then $\bar{\Phi}^n f(x; *; 0, ..., 0) : X^{\otimes n} \to Y$ is a $\mathbf{K}$ n-linear $C^0(U, L_n(X^{\otimes n}, Y))$ symmetric map.*

Proof. From Definition 1 it follows, that $f^{[1]}(x, v, t) = \bar{\Phi}^1 f(x; v; t)$ on $U^{[1]} = U^{(1)}$, so both $\mathbf{K}$-linear spaces are linearly topologically isomorphic. On the other hand, it was proved in Proposition 2.2 and Lemma 4.8 [1], that $\bar{\Phi}^n f(x; *, 0, ..., 0)$ is the $\mathbf{K}$ n-linear symmetric mapping for each $x \in U$ and it belongs to $C^0(U, L_n(X^{\otimes n}, Y))$, since $\bar{\Phi}^n f(x; v_1, ..., v_n; t_1, ..., t_n)$ is continuous on $U^{(n)}$ and for each $x \in U$ and $v_1, ..., v_n \in X$ there exist neighborhoods V_i of v_i in X and W of zero in $\mathbf{K}$ such that $x + WV_1 + ... + WV_n \subset U$.

3. Lemma. *Operators $\Upsilon^n(f) := f^{[n]}$ from $C^{[n]}(U, Y)$ into $C^0(U^{[n]}, Y)$ and $\bar{\Phi}^n : C^n(U, Y) \to C^0(U^{(n)}, Y)$ are $\mathbf{K}$-linear and continuous.*

Proof. Since $[(af + bg)(x + vt) - (af + bg)(x)]/t = a(f(x + vt) - f(x))/t+$

$b(g(x+vt)-g(x))/t$ for each $f,g\in C^1(U,Y)$ and each $a,b\in \mathbf{K}$, then applying this formula by induction and using definitions of operators Υ^n and $\bar\Phi^n$ we get their $\mathbf{K}$-linearity. Indeed,

$\Upsilon^n(af+bg)(x^{[n]})=\Upsilon^1(\Upsilon^{n-1}(af+bg)(x^{[n-1]}))(x^{[n]})=\Upsilon^1(af^{[n-1]}+bg^{[n-1]})(x^{[n]})= af^{[n]}(x^{[n]})+bg^{[n]}(x^{[n]})$ and

$\bar\Phi^n(af+bg)(x^{(n)})=\bar\Phi^1(\bar\Phi^{n-1}(af+bg)(x^{(n-1)}))(x^{(n)})=\bar\Phi^1(af^{(n-1)}+bg^{(n-1)})(x^{(n)})=af^{(n)}(x^{(n)})+bg^{(n)}(x^{(n)})$.

The continuity of Υ^n and $\bar\Phi^n$ follows from definitions of spaces $C^{[n]}(U,Y)$ and $C^n(U,Y)$ respectively.

4. Lemma. *Let either $f,g\in C^{[n]}(U,Y)$, where U is an open subset in X, Y is an algebra over $\mathbf{K}$, or $f\in C^{[n]}(U,\mathbf{K})$ and $g\in C^{[n]}(U,Y)$, where Y is a topological vector space over $\mathbf{K}$, then*

(1) $(fg)^{[n]}(x^{[n]})=(\Upsilon\otimes\hat P+\hat\pi\otimes\Upsilon)^n.(f\otimes g)(x^{[n]})$

and $(fg)^{[n]}\in C^0(U^{[n]},Y)$, where $(\hat\pi^k g)(x^{[k]}):=g\circ\pi_1^0\circ\pi_2^1\circ...\circ\pi_k^{k-1}(x^{[k]})$, $\hat P^n g:=P_nP_{n-1}...P_1g$, $\pi_k^{k-1}(x^{[k]}):=x^{[k-1]}$, $(A\otimes B).(f\otimes g):=(Af)(Bg)$ for $A,B\in L(C^n(U,Y),C^m(U,Y))$, $m\le n$, $(A_1\otimes B_1)...(A_k\otimes B_k).(f\otimes g):=(A_1...A_k\otimes B_1...B_k).(f\otimes g):=(A_1...A_kf)(B_1...B_kg)$ for corresponding operators, $\Upsilon^n f:=f^{[n]}$, $(P_kg)(x^{[k]}):=g(x^{[k-1]}+v^{[k-1]}t_k)$, $\hat P^k\hat\pi^{a_1}\Upsilon^{b_1}...\hat\pi^{a_l}\Upsilon^{b_l}g=P_{k+s}...P_{s+1}\hat\pi^{a_1}\Upsilon^{b_1}...\hat\pi^{a_l}\Upsilon^{b_l}g$ with $s=b_1+...+b_l-a_1-...-a_l\ge 0$, $a_1,...,a_l,b_1,...,b_l\in\{0,1,2,3,...\}$.

Proof. Let at first $n=1$, then

(2) $(fg)^{[1]}(x^{[1]})=[(fg)(x+vt)-(fg)(x)]/t=[(f(x+vt)-f(x))g(x+vt)+f(x)(g(x+vt)-g(x))]/t=(\Upsilon^1f)(x^{[1]})(P_1g)(x^{[1]})+(\hat\pi_1^0f)(x^{[1]})\Upsilon^1g(x^{[1]})$,

since $\hat\pi_1^0(x^{[1]})=x$ and P_1 is the composition of the projection $\hat\pi_1^0$ and the shift operator on vt. Let now $n=2$, then applying Formula (2) we get:

(3) $(fg)^{[2]}(x^{[2]})=((fg)^{[1]}(x^{[1]}))^{[1]}(x^{[2]})=(\Upsilon^1(f^{[1]}(x^{[1]})(x^{[2]}))g(x+(v^{[0]}+v_2^{[1]}t_2)(t_1+v_3^{[1]}t_2)+v_1^{[1]}t_2)+f^{[1]}(x^{[1]})g^{[1]}(x+v^{[0]}t_1,v_1^{[1]}+v_2^{[1]}(t_1+v_3^{[1]}t_2),t_2)+f^{[1]}(x,v_1^{[1]},t_2)g^{[1]}(x^{[1]}+v_1^{[1]}t_2)+f(x)g^{[2]}(x^{[2]})$,

where $v^{[k]}=(v_1^{[k]},v_2^{[k]},v_3^{[k]})$ for each $k\ge 1$ and $v^{[0]}=v_1^{[0]}$ such that $x^{[k]}+v^{[k]}t_{k+1}=(x^{[k]}+v_1^{[k]}t_{k+1},v^{[k-1]}+v_2^{[k]}t_{k+1},t_k+v_3^{[k]}t_{k+1})$ for each $1\le k\in\mathbf{Z}$.
For $n=3$ we get

(4) $(fg)^{[3]}(x^{[3]})=[(\Upsilon^3f)(\hat P^3g)+(\hat\pi^1\Upsilon^2f)(\Upsilon^1\hat P^2g)+(\Upsilon^1(\hat\pi^1\Upsilon^1f))(\hat P^1\Upsilon^1\hat P^1g)+(\hat\pi^2\Upsilon^1f)(\Upsilon^2\hat P^1g)+(\Upsilon^2\hat\pi^1f)(\hat P^2\Upsilon^1g)+(\hat\pi^1\Upsilon^1\hat\pi^1f)(\Upsilon^1\hat P^1\Upsilon^1g)+(\Upsilon^1(\hat\pi^2f))(\hat P^1\Upsilon^2g)+(\hat\pi^3f)(\Upsilon^3g)](x^{[3]})$,

since by our definition $\hat P^k\hat\pi^{a_1}\Upsilon^{b_1}...\hat\pi^{a_l}\Upsilon^{b_l}g=P_{k+s}...P_{s+1}\hat\pi^{a_1}\Upsilon^{b_1}...\hat\pi^{a_l}\Upsilon^{b_l}g$ with $s=b_1+...+b_l-a_1-...-a_l\ge 0$, $a_1,...,a_l,b_1,...,b_l\in\{0,1,2,3,...\}$.

Therefore, Formula (1) for $n=1$ and $n=2$ and $n=3$ is demonstrated by Formulas $(2-4)$. If $f,g\in C^0(U^{[k]},Y)$, $a,b\in \mathbf{K}$, then $(P_k(af+bg))(x^{[k]}) := (af+bg)(x^{[k-1]}+v^{[k-1]}t_k) = af(x^{[k-1]}+v^{[k-1]}t_k)+bg(x^{[k-1]}+v^{[k-1]}t_k)$, moreover, $\hat{\pi}^k(af+bg)(x^{[k]}) = (af+bg)\circ\pi_1^0\circ\pi_2^1\circ...\circ\pi_k^{k-1}(x^{[k]}) = (af+bg)(x) = af(x)+bg(x) = a\hat{\pi}^k f(x^{[k]})+b\hat{\pi}^k g(x^{[k]})$ for each $x^{[k]}\in U^{[k]}$, hence $\hat{\pi}^k$ and P_k and $\hat{P}^k$ are $\mathbf{K}$-linear operators for each $k\in\mathbf{N}$. Suppose that Formula (1) is proved for $n=1,...,m$, then for $n=m+1$ it follows by application of Formula (2) to both sides of Formula (1) for $n=m$:

$(fg)^{m+1}(x^{[m+1]}) = ((fg)^{[m]}(x^{[m]}))^{[1]}(x^{[m+1]}) = ((\Upsilon\otimes\hat{P}+\hat{\pi}\otimes\Upsilon)^m.(f\otimes g)(x^{[m]}))^{[1]}(x^{[m+1]}) = (\Upsilon\otimes\hat{P}+\hat{\pi}\otimes\Upsilon)^{m+1}.(f\otimes g)(x^{[m+1]})$,
since $x^{[m+1]}=(x^{[m]})^{[1]}$ and more generally $x^{[m+k]}=(x^{[m]})^{[k]}$ for each nonnegative integers m and k such that $\pi_k^{k-1}(x^{[m+k]}) = x^{[m+k-1]}$ for $k\ge 1$; Υ^k, $\hat{P}^k$ and $\hat{\pi}$ are $\mathbf{K}$-linear operators on corresponding spaces of functions (see above and Lemma 3) and

$(\Upsilon\otimes\hat{P}+\hat{\pi}\otimes\Upsilon)^{m+1}.(f\otimes g)(x^{[m+1]}) =$
$\sum_{a_1+...+a_{m+1}+b_1+...+b_{m+1}=m+1}(\Upsilon^{a_1}\otimes\hat{P}^{a_1})$
$(\hat{\pi}^{b_1}\otimes\Upsilon^{b_1})...(\Upsilon^{a_{m+1}}\otimes\hat{P}^{a_{m+1}})(\hat{\pi}^{b_{m+1}}\otimes\Upsilon^{b_{m+1}}).(f\otimes g)(x^{[m+1]})$,
where a_j and b_j are nonnegative integers for each $j=1,...,m+1$, $(A_1\otimes B_1)...(A_k\otimes B_k).(f\otimes g) := (A_1...A_k\otimes B_1...B_k).(f\otimes g) := (A_1...A_kf)(B_1...B_kg)$.

5. Note. Consider the projection

(1) $\psi_n : X^{m(n)}\times\mathbf{K}^{s(n)}\to X^{l(n)}\times\mathbf{K}^n$,

where $m(n)=2m(n-1)$, $s(n)=2s(n-1)+1$, $l(n)=n+1$ for each $n\in\mathbf{N}$ such that $m(0)=1$, $s(0)=0$, $m(n)=2^n$, $s(n)=1+2+2^2+...+2^{n-1}=2^n-1$. Then $m(n)$, $s(n)$, $l(n)$ and n correspond to number of variables in X, $\mathbf{K}$ for Υ^n, in X and $\mathbf{K}$ for $\bar{\Phi}^n$ respectively. Therefore, $\psi(x^{[n]}) = x^{(n)}$ and $\psi_n(U^{[n]}) = U^{(n)}$ for each $n\in\mathbf{N}$ for suitable ordering of variables. Thus $\bar{\Phi}^n f(x^{(n)}) = \hat{\psi}_n\Upsilon^n f(x^{[n]}) = f^{[n]}(x^{[n]})|_{W^{(n)}}$, where $\hat{\psi}_n g(y) := g(\psi_n(y))$ for a function g on a subset V in $X^{l(n)}\times\mathbf{K}^n$ for each $y\in\psi_n^{-1}(V)\subset X^{m(n)}\times\mathbf{K}^{s(n)}$, $W^{(n)} = U^{(n)}\times 0$, $0\in X^{m(n)-l(n)}\times\mathbf{K}^{s(n)-n}$ for the corresponding ordering of variables.

6. Corollary. *Let either $f,g\in C^n(U,Y)$, where U is an open subset in X, Y is an algebra over $\mathbf{K}$, or $f\in C^n(U,\mathbf{K})$ and $g\in C^n(U,Y)$, where Y is a topological vector space over $\mathbf{K}$, then*

(1) $\bar{\Phi}^n(fg)(x^{(n)}) = (\bar{\Phi}\otimes\hat{P}+\hat{\pi}\otimes\bar{\Phi})^n.(f\otimes g)(x^{(n)})$

and $\bar{\Phi}^n(fg)\in C^0(U^{(n)},Y)$. In more details:

(2) $\bar{\Phi}^n(fg)(x^{(n)}) = \sum_{0\le a,0\le b,a+b=n}\sum_{j_1<...<j_a;s_1<...<s_b;\{j_1,...,j_a\}\cup\{s_1,...,s_b\}=\{1,...,n\}}$

$\bar{\Phi}^a f(x; v_{j_1}, ..., v_{j_a}; t_{j_1}, ..., t_{j_a}) \bar{\Phi}^b g(x + v_{j_1} t_{j_1} + ... + v_{j_a} t_{j_a}; v_{s_1}, ..., v_{s_b}; t_{s_1}, ..., t_{s_b})$.

Proof. The operator $\hat{\psi}_n$ is $\mathbf{K}$-linear, since $\hat{\psi}_n(af + bg)(y) = (af + bg)(\psi_n(y)) = af(\psi_n(y)) + bg(\psi_n(y))$ for each $a, b \in \mathbf{K}$ and functions f, g on a subset V in $X^{l(n)} \times \mathbf{K}^n$ and each $y \in \psi_n^{-1}(V) \subset X^{m(n)} \times \mathbf{K}^{s(n)}$. Mention that the restrictions of $\hat{\pi}_k^{k-1}$ and P_k on $W^{(k)}$ gives $\pi_k^{k-1}(x^{(k)}) := x^{(k-1)}$ and $(P_k g)(x^{(k)}) := g(x^{(k-1)} + v_k t_k)$ in the notation of §1. The application of the operator $\hat{\psi}_n$ to both sides of Equation 4(1) gives Equation (1) of this corollary, since $\hat{\psi}_n \Upsilon^n = \bar{\Phi}^n$ for each nonnegative integer n, where $\Upsilon^0 = I$ and $\bar{\Phi}^0 = I$ and $\hat{\psi}_0 = I$ are the unit operators.

7. Lemma. *Let $f_1, ..., f_k \in C^{[n]}(U, Y)$, where U is an open subset in X, either Y is an algebra over $\mathbf{K}$, or $f_1, ..., f_{k-1} \in C^{[n]}(U, \mathbf{K})$ and $f_k \in C^{[n]}(U, Y)$, where Y is a topological vector space over $\mathbf{K}$, then*

(1) $(f_1 ... f_k)^{[n]}(x^{[n]}) = [\sum_{\alpha=0}^{k-1} \hat{\pi}^{\otimes \alpha} \otimes \Upsilon \otimes \hat{P}^{\otimes (k-\alpha-1)}]^n . (f_1 \otimes ... \otimes f_k)(x^{[n]})$ *and* $(f_1 ... f_k)^{[n]} \in C^0(U^{[n]}, Y)$, *where* $\hat{\pi}^{\otimes \alpha} \otimes \Upsilon \otimes \hat{P}^{\otimes (k-\alpha-1)} . (f_1 \otimes ... \otimes f_k) := (\hat{\pi}(f_1 ... f_\alpha))(\Upsilon f_{\alpha+1})(\hat{P}(f_{\alpha+2} ... f_k))$, *where* $\hat{\pi}^0 := I$, $\hat{P}^0 = I$ *is the unit operator,* $\hat{\pi} f_0 := 1$, $\hat{P} f_{k+1} := 1$ *(see Lemma 4).*

Proof. Consider at first $n = 1$ and apply Formula 4(1) by induction to appearing products of functions, then

(2) $\Upsilon^1(f_1 ... f_k)(x^{[1]}) = [(\Upsilon^1(f_1 ... f_{k-1}))(P_1 f_k) + (\hat{\pi}^1(f_1 ... f_{k-1}))(\Upsilon^1 f_k)](x^{[1]}) =$
$[(\Upsilon^1(f_1 ... f_{k-2}))(P_1 f_{k-1})(P_1 f_k) + (\hat{\pi}^1(f_1 ... f_{k-2}))(\Upsilon^1 f_{k-1})(P_1 f_k)$
$+ (\hat{\pi}^1(f_1 ... f_{k-1}))(\Upsilon^1 f_k)](x^{[1]}) = ...$
$= (\sum_{\alpha=0}^{k-1} (\hat{\pi}^1)^{\otimes \alpha} \otimes \Upsilon^1 \otimes P_1^{\otimes (k-\alpha-1)}) . (f_1 \otimes ... \otimes f_k)$,
where $A^{\otimes \alpha} \otimes B \otimes C^{\otimes (k-\alpha-1)} . (f_1 \otimes ... \otimes f_k) := (A(f_1 ... f_\alpha))(B f_{\alpha+1})(C(f_{\alpha+2} ... f_k))$ for operators A, B and C and each nonnegative integer α, where $A^0 := I$, $C^0 = I$ is the unit operator, $A f_0 := 1$, $C f_{k+1} := 1$, in particular, $A = \hat{\pi}^1$, $B = \Upsilon^1$, $C = P_1$. Thus, acting by induction on both sides by Υ^1 from Formula (2) we get Formula (1) of this lemma, since the product of n terms $\Upsilon^1 ... \Upsilon^1$ is equal to Υ^n.

8. Corollary. *Let $f_1, ..., f_k \in C^n(U, Y)$, where U is an open subset in X, either Y is an algebra over $\mathbf{K}$, or $f_1, ..., f_{k-1} \in C^n(U, \mathbf{K})$ and $f_k \in C^n(U, Y)$, where Y is a topological vector space over $\mathbf{K}$, then*

(1) $\bar{\Phi}^n(f_1 ... f_k)(x^{(n)}) = [\sum_{\alpha=0}^{k-1} \hat{\pi}^{\otimes \alpha} \otimes \bar{\Phi} \otimes \hat{P}^{\otimes (k-\alpha-1)}]^n . (f_1 \otimes ... \otimes f_k)(x^{(n)})$ *and* $\bar{\Phi}^n(f_1 ... f_k) \in C^0(U^{(n)}, Y)$, *where* $\hat{\pi}^{\otimes \alpha} \otimes \bar{\Phi} \otimes \hat{P}^{\otimes (k-\alpha-1)} . (f_1 \otimes ... \otimes f_k) := (\hat{\pi}(f_1 ... f_\alpha))(\bar{\Phi} f_{\alpha+1})(\hat{P}(f_{\alpha+2} ... f_k))$ *(see Lemma 7).*

Proof. Applying operator $\hat{\psi}_n$ from Note 5 to both sides of Equation 7(1) we get Formula (1) of this Corollary.

9. Lemma. *Let $u \in C^{[n]}(\mathbf{K}^s, \mathbf{K}^m)$, $u(\mathbf{K}^s) \subset U$ and $f \in C^{[n]}(U, Y)$, where U is an open subset in $\mathbf{K}^m$, $s, m \in \mathbf{N}$, Y is a $\mathbf{K}$-linear space, then*

(1) $(f \circ u)^{[n]}(x^{[n]}) = [\sum_{j_1=1}^m ... \sum_{j_n=1}^{m(n)} (A_{j_n, v^{[n-1]}, t_n} ... A_{j_1, v^{[0]}, t_1} f \circ u)(\Upsilon^1 \circ p_{j_n} \hat{S}_{j_{n-1}+1, v^{[n-2]} t_{n-1}} ... \hat{S}_{j_1+1, v^{[0]} t_1} u^{n-1})(P_n \Upsilon^1 \circ p_{j_{n-1}} \hat{S}_{j_{n-2}+1, v^{[n-3]} t_{n-2}} ... \hat{S}_{j_1+1, v^{[0]} t_1} u^{n-2}) ... (P_n ... P_2 \Upsilon^1 \circ p_{j_1} u) + \sum_{j_1=1}^m ... \sum_{j_{n-1}=1}^{m(n-1)} (\hat{\pi}^1 (A_{j_{n-1}, v^{[n-2]}, t_{n-1}} ... A_{j_1, v^{[0]}, t_1} f \circ u) [\sum_{\alpha=0}^{n-2} \hat{\pi}^{\otimes \alpha} \otimes \Upsilon \otimes \hat{P}^{\otimes (n-\alpha-2)}]((\Upsilon^1 \circ p_{j_{n-1}} \hat{S}_{j_{n-2}+1, v^{[n-3]} t_{n-2}} ... \hat{S}_{j_1+1, v^{[0]} t_1} u^{n-2}) \otimes ... \otimes (P_{n-1} ... P_2 \Upsilon^1 \circ p_{j_1} u))$
$+ [\sum_{\alpha=0}^{n-2} \hat{\pi}^{\otimes \alpha} \otimes \Upsilon \otimes \hat{P}^{\otimes (n-\alpha-2)}](\sum_{j_1=1}^m ... \sum_{j_{n-2}=1}^{m(n-2)} (\hat{\pi}^1 (A_{j_{n-2}, v^{[n-3]}, t_{n-2}} ... A_{j_1, v^{[0]}, t_1} f \circ u)) \otimes [\sum_{\alpha=0}^{n-3} \hat{\pi}^{\otimes \alpha} \otimes \Upsilon \otimes \hat{P}^{\otimes (n-\alpha-3)}]((\Upsilon^1 \circ p_{j_{n-2}} \hat{S}_{j_{n-3}+1, v^{[n-4]} t_{n-3}} ... \hat{S}_{j_1+1, v^{[0]} t_1} u^{n-3}) \otimes ... \otimes (P_{n-2} ... P_2 \Upsilon^1 \circ p_{j_1} u)) + ...$
$+ [\sum_{\alpha=0}^2 \hat{\pi}^{\otimes \alpha} \otimes \Upsilon \otimes \hat{P}^{\otimes (2-\alpha)}]^{n-3} \{ \sum_{j_1=1}^m \sum_{j_2=1}^{m(2)} (\hat{\pi}^1 A_{j_2, v^{[1]}, t_2} A_{j_1, v^{[0]}, t_1} f \circ u)(\Upsilon^1 \otimes \hat{P}^1 + \hat{\pi}^1 \otimes \Upsilon^1)((\Upsilon^1 \circ p_{j_2} \hat{S}_{j_1+1, v^{[0]} t_1} u) \otimes (P_2 \Upsilon^1 \circ p_{j_1} u)) \}$
$+ (\Upsilon \otimes \hat{P} + \hat{\pi} \otimes \Upsilon)^{n-2} \{ \sum_{j_1=1}^m (\hat{\pi}^1 A_{j_1, v^{[0]}, t_1} f \circ u) \otimes (\Upsilon^2 \circ p_{j_1} u) \}](x^{[n]})$
and $f \circ u \in C^0((\mathbf{K}^s)^{[n]}, Y)$, where $S_{j,\tau} u(y) := (u_1(y), ..., u_{j-1}(y), u_j(y + \tau_{(s)}), u_{j+1}(y + \tau_{(s)}), ..., u_m(y + \tau_{(s)}))$, $u = (u_1, ..., u_m)$, $u_j \in \mathbf{K}$ for each $j = 1, ..., m$, $y \in \mathbf{K}^s$, $\tau = (\tau_1, ..., \tau_k) \in \mathbf{K}^k$, $k \ge s$, $\tau_{(s)} := (\tau_1, ..., \tau_s)$, $p_j(x) := x_j$, $x = (x_1, ..., x_m)$, $x_j \in \mathbf{K}$ for each $j = 1, ..., m$, $\hat{S}_{j+1,\tau} g(u(y), \beta) := g(S_{j+1,\tau} u(y), \beta)$, $y \in \mathbf{K}^s$, β is some parameter, $A_{j,v,t} := (\hat{S}_{j+1, vt} \otimes t \Upsilon^1 \circ p_j)^* \Upsilon^1_j$, where Υ^1 is taken for variables (x, v, t) or corresponding to them after actions of preceding operations as Υ^k, $\Upsilon^1_j f(x, v_j, t) := [f(x + e_j v_j t) - f(x)]/t$, $(B \otimes A)^* \Upsilon^1 f_i \circ u^i(x, v, t) := \Upsilon^1_j f_i(B u^i, v, A u^i)$, $B : \mathbf{K}^{m(i)} \to \mathbf{K}^{m(i)}$, $A : \mathbf{K}^{m(i)} \to \mathbf{K}$, $e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathbf{K}^{m(i)}$ with 1 on j-th place; $m(i) = m + i - 1$, $j_i = 1, ..., m(i)$; $u^1 := u$, $u^2 := (u^1, t_1 \Upsilon^1 \circ p_{j_1} u^1)$,..., $u^n = (u^{n-1}, t_{n-1} \Upsilon^1 \circ p_{j_{n-1}} u^{n-1})$, $A_{j_1, v^{[0]}, t_1} f \circ u =: f_1 \circ u^1$, $A_{j_n, v^{[n-1]}, t_n} f_{n-1} \circ u^{n-1} =: f_n \circ u^n$, $\hat{S}_* \Upsilon^1 f(z) := \Upsilon^1 f(\hat{S}_* z)$.

Proof. At first consider $n = 1$, then $(f \circ u)^{[1]}(t_0, v, t) = [f(u(t_0 + vt)) - f(u(t_0))]/t$, where $t_0 \in \mathbf{K}^s$, $t \in \mathbf{K}$, $v \in \mathbf{K}^s$. Though we consider here the general case mention, that in the particular case $s = 1$ one has $t_0 \in \mathbf{K}$, $v \in \mathbf{K}$. Then

$(f \circ u)^{[1]}(t_0, v, t) = [f(u(t_0 + vt)) - f(u_1(t_0), u_2(t_0 + vt), ..., u_m(t_0 + vt))]/t + [f(u_1(t_0), u_2(t_0 + vt), u_3(t_0 + vt), ..., u_m(t_0 + vt)) - f(u_1(t_0), u_2(t_0), u_3(t_0 + vt), ..., u_m(t_0 + vt))]/t + ... + [f(u_1(t_0), ..., u_{m-1}(t_0), u_m(t_0 + vt)) - f(u(t_0))]/t$, where $u = (u_1, ..., u_m)$, $u_j \in \mathbf{K}$ for each $j = 1, ..., m$. Since $u_j(t_0 + vt) - u_j(t_0) = t u_j^{[1]}(t_0, v, t)$, hence

$(f\circ u)^{[1]}(t_0,v,t)=\Upsilon^1 f((u_1(t_0),u_2(t_0+vt),...,u_m(t_0+vt)),e_1,t\Upsilon^1 u_1(t_0,v,t))$ $\Upsilon^1 u_1(t_0,v,t)+\Upsilon^1 f((u_1(t_0),u_2(t_0),u_3(t_0+vt),...,u_m(t_0+vt)),e_2,t\Upsilon^1 u_2(t_0,v,t))$ $\Upsilon^1 u_2(t_0,v,t)+...+\Upsilon^1 f(u(t_0),e_m,t\Upsilon^1 u_m(t_0,v,t))\Upsilon^1 u_m(t_0,v,t)$,
since $u_j\in\mathbf{K}$ for each $j=1,...,m$ and $\mathbf{K}$ is the field, where $e_j=(0,...,0,1,0,...,0)\in\mathbf{K}^m$ with 1 on j-th place for each $j=1,...,m$. With the help of shift operators it is possible to write the latter formula shorter:

(2) $\Upsilon^1(f\circ u)(y,v,t)=\sum_{j=1}^m \hat{S}_{j+1,vt}\Upsilon^1 f(u(y),e_j,t\Upsilon^1\circ p_j u(y,v,t))(\Upsilon^1\circ p_j u(y,v,t))$,

where $p_j(x):=x_j$, $x=(x_1,...,x_m)$, $x_j\in\mathbf{K}$ for each $j=1,...,m$, $\hat{S}_{j+1,\tau}g(u(y),\beta):=g(S_{j+1,\tau}u(y),\beta)$, $y\in\mathbf{K}^s$, $\tau\in\mathbf{K}^k$, $k\ge s$, β is some parameter. Introduce operators $A_{j,v,t}:=(\hat{S}_{j+1,vt}\otimes t\Upsilon^1\circ p_j)^*\Upsilon^1_j$, where Υ^1 is taken for variables (y,v,t) or corresponding to them after actions of preceding operators as Υ^k remembering that $y^{[k]},v^{[k]}\in(\mathbf{K}^s)^{[k]}$, $t\in\mathbf{K}$, $v^{[k]}=(v_1^{[k]},v_2^{[k]},v_3^{[k]})$ with $v_1^{[k]},v_2^{[k]}\in(\mathbf{K}^s)^{[k-1]}$, $v_3^{[k]}\in\mathbf{K}^k$ for each $k\ge 1$, in particular, $v^{[0]}=v_1^{[0]}$ for $k=0$, $\Upsilon^1_j f(x,v,t):=[f(x+e_jv_jt)-f(x)]/t$, $(B\otimes A)^*\Upsilon^1 f_i\circ u^i(y,v,t):=\Upsilon^1_j f_i(Bu^i,v,Au^i)$, $B:\mathbf{K}^{m(i)}\to\mathbf{K}^{m(i)}$, $A:\mathbf{K}^{m(i)}\to\mathbf{K}$. For example, in the particular case of $s=1$ we have $v^{[k]}\in(\mathbf{K})^{[k]}$. Therefore, in the general case Formula (2) takes the form:

(3) $\Upsilon^1 f\circ u(y,v,t)=\sum_{j=1}^m(A_{j,v,t}f\circ u)(\Upsilon^1\circ p_j u)(y,v,t)$.

Take now $n=2$, then

$\Upsilon^2 f\circ u(y^{[2]})=\Upsilon^1\sum_{j=1}^m[(A_{j,v,t}f\circ u)(\Upsilon^1\circ p_j u)(y,v,t)](y^{[2]})$.

In the square brackets there is the product, hence from Formula 4(1) and Lemma 3 we get:

(4) $\Upsilon^2 f\circ u(y^{[2]})=\sum_{j=1}^m[(\Upsilon^1 A_{j,v^{[0]},t}f\circ u)(P_2\Upsilon^1\circ p_j u)+(\hat{\pi}^1 A_{j,v^{[0]},t}f\circ u)(\Upsilon^2\circ p_j u)](y^{[2]})$.

Then from Formula (3) applied to terms $A_{j,v,t}f\circ u$ it follows, that $\Upsilon^1 A_{j_1,v^{[0]},t_1}f\circ u(y^{[2]})=\sum_{j_2=1}^{m(2)}(A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1}f\circ u)(\Upsilon^1\circ p_{j_2}S_{j_1+1,v^{[0]}t_1}u)(y^{[2]})$, where $v^{[0]}=v$, $t_1=t$ (see also Lemma 4). Therefore,

(5) $\Upsilon^2 f\circ u(y^{[2]})=[\sum_{j_1=1}^m\sum_{j_2=1}^{m(2)}(A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1}f\circ u)(\Upsilon^1\circ p_{j_2}\hat{S}_{j_1+1,v^{[0]}t_1}u)(P_2\Upsilon^1\circ p_{j_1}u)+\sum_{j_1=1}^m(\hat{\pi}^1 A_{j_1,v^{[0]},t_1}f\circ u)(\Upsilon^2\circ p_{j_1}u)](y^{[2]})$.

Then for $n=3$ applying Formulas (3) and 7(1) to (5) we get:

(6) $\Upsilon^3 f\circ u(y^{[3]})=[\sum_{j_1=1}^m\sum_{j_2=1}^{m(2)}\sum_{j_3=1}^{m(3)}(A_{j_3,v^{[2]},t_3}A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1}f\circ u)$ $(\Upsilon^1\circ p_{j_3}\hat{S}_{j_2+1,v^{[1]}t_2}\hat{S}_{j_1+1,v^{[0]}t_1}u^2)(P_2\Upsilon^1\circ p_{j_2}\hat{S}_{j_1+1,v^{[0]}t_1}u)(P_3P_2\Upsilon^1\circ p_{j_1}u)+$ $\sum_{j_1=1}^m\sum_{j_2=1}^{m(2)}[(\hat{\pi}^1(A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1}f\circ u))(\Upsilon^2\circ p_{j_2}\hat{S}_{j_1+1,v^{[0]}t_1}u)(P_3P_2\Upsilon^1\circ p_{j_1}u)+$ $(\hat{\pi}^1\{(A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1}f\circ u)(\Upsilon^1\circ p_{j_2}\hat{S}_{j_1+1,v^{[0]}t_1}u)\}(\Upsilon^1P_2\Upsilon^1\circ p_{j_1}u)]+$

$\sum_{j_1=1}^m \sum_{j_3=1}^{m(3)}(A_{j_3,v^{[2]},t_3}\hat{\pi}^1 A_{j_1,v^{[0]},t_1} f\circ u)(\Upsilon^1\circ p_{j_3}\hat{S}_{j_1+1,v^{[0]}t_1}u)(P_3\Upsilon^2\circ p_{j_1}u)$
$+\sum_{j_1=1}^m(\hat{\pi}^2 A_{j_1,v^{[0]},t_1} f\circ u)(\Upsilon^3\circ p_{j_1}u)](y^{[3]})$.

Thus Formula (1) is proved for $n=1,2,3$. Suppose that it is true for $k=1,...,n$ and prove it for $k=n+1$. Applying Formula 7(1) to both sides of (1) we get:

(7) $\Upsilon^{n+1}f\circ u(y^{[n+1]})=[\sum_{j_1=1}^m ... \sum_{j_{n+1}=1}^{m(n+1)}(A_{j_{n+1},v^{[n]},t_{n+1}}...A_{j_1,v^{[0]},t_1} f\circ u)$
$(\Upsilon^1\circ p_{j_{n+1}}\hat{S}_{j_n+1,v^{[n-1]}t_n}...\hat{S}_{j_1+1,v^{[0]}t_1}u^n)(P_{n+1}\Upsilon^1\circ p_{j_n}\hat{S}_{j_{n-1}+1,v^{[n-2]}t_{n-1}}...\hat{S}_{j_1+1,v^{[0]}t_1}u^{n-1})...$
$(P_{n+1}...P_2\Upsilon^1\circ p_{j_1}u)+\sum_{j_1=1}^m ... \sum_{j_n=1}^{m(n)}(\hat{\pi}^1(A_{j_n,v^{[n-1]},t_n}...A_{j_1,v^{[0]},t_1} f\circ u)$
$\Upsilon^1((\Upsilon^1\circ p_{j_n}\hat{S}_{j_{n-1}+1,v^{[n-2]}t_{n-1}}...\hat{S}_{j_1+1,v^{[0]}t_1}u^{n-1})...(P_n...P_2\Upsilon^1\circ p_{j_1}u))+$
$\Upsilon^1(\sum_{j_1=1}^m ... \sum_{j_{n-1}=1}^{m(n-1)}(\hat{\pi}^1(A_{j_{n-1},v^{[n-2]},t_{n-1}}...A_{j_1,v^{[0]},t_1} f\circ u))\Upsilon^1((\Upsilon^1\circ p_{j_{n-1}}\hat{S}_{j_{n-2}+1,v^{[n-3]}t_{n-2}}$
$...\hat{S}_{j_1+1,v^{[0]}t_1}u^{n-2})...(P_{n-1}...P_2\Upsilon^1\circ p_{j_1}u))+...+\Upsilon^{n-2}\{\sum_{j_1=1}^m\sum_{j_2=1}^{m(2)}$
$(\hat{\pi}^1 A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1} f\circ u)\Upsilon^1((\Upsilon^1\circ p_{j_2}\hat{S}_{j_1+1,v^{[0]}t_1}u)(P_2\Upsilon^1\circ p_{j_1}u))\}+$
$\Upsilon^{n-1}\{\sum_{j_1=1}^m\hat{\pi}^1 A_{j_1,v^{[0]},t_1} f\circ u)(\Upsilon^2\circ p_{j_1}u)\}](y^{[n+1]})=$
$[\sum_{j_1=1}^m ... \sum_{j_{n+1}=1}^{m(n+1)}(A_{j_{n+1},v^{[n]},t_{n+1}}...A_{j_1,v^{[0]},t_1} f\circ u)(\Upsilon^1\circ p_{j_{n+1}}\hat{S}_{j_n+1,v^{[n-1]}t_n}$
$...\hat{S}_{j_1+1,v^{[0]}t_1}u^n)(P_{n+1}\Upsilon^1\circ p_{j_n}\hat{S}_{j_{n-1}+1,v^{[n-2]}t_{n-1}}...\hat{S}_{j_1+1,v^{[0]}t_1}u^{n-1})...(P_{n+1}...P_2\Upsilon^1\circ$
$p_{j_1}u)+\sum_{j_1=1}^m ... \sum_{j_n=1}^{m(n)}(\hat{\pi}^1(A_{j_n,v^{[n-1]},t_n}...A_{j_1,v^{[0]},t_1} f\circ u)[\sum_{\alpha=0}^{n-1}\hat{\pi}^{\otimes\alpha}\otimes\Upsilon\otimes\hat{P}^{\otimes(n-\alpha-1)}]((\Upsilon^1\circ$
$p_{j_n}\hat{S}_{j_{n-1}+1,v^{[n-2]}t_{n-1}}...\hat{S}_{j_1+1,v^{[0]}t_1}u^{n-1})\otimes...\otimes(P_n...P_2\Upsilon^1\circ p_{j_1}u))$
$+[\sum_{\alpha=0}^{n-1}\hat{\pi}^{\otimes\alpha}\otimes\Upsilon\otimes\hat{P}^{\otimes(n-\alpha-1)}](\sum_{j_1=1}^m ... \sum_{j_{n-1}=1}^{m(n-1)}(\hat{\pi}^1(A_{j_{n-1},v^{[n-2]},t_{n-1}}...A_{j_1,v^{[0]},t_1} f\circ$
$u))\otimes[\sum_{\alpha=0}^{n-2}\hat{\pi}^{\otimes\alpha}\otimes\Upsilon\otimes\hat{P}^{\otimes(n-\alpha-2)}]((\Upsilon^1\circ p_{j_{n-1}}\hat{S}_{j_{n-2}+1,v^{[n-3]}t_{n-2}}...\hat{S}_{j_1+1,v^{[0]}t_1}u^{n-2})\otimes$
$...\otimes(P_{n-1}...P_2\Upsilon^1\circ p_{j_1}u))$
$+[\sum_{\alpha=0}^2\hat{\pi}^{\otimes\alpha}\otimes\Upsilon\otimes\hat{P}^{\otimes(2-\alpha)}]^{n-2}\{\sum_{j_1=1}^m\sum_{j_2=1}^{m(2)}(\hat{\pi}^1 A_{j_2,v^{[1]},t_2}A_{j_1,v^{[0]},t_1} f\circ u)(\Upsilon^1\otimes$
$\hat{P}^1+\hat{\pi}^1\otimes\Upsilon^1)((\Upsilon^1\circ p_{j_2}\hat{S}_{j_1+1,v^{[0]}t_1}u)\otimes(P_2\Upsilon^1\circ p_{j_1}u))\}$
$+(\Upsilon\otimes\hat{P}+\hat{\pi}\otimes\Upsilon)^{n-1}\{\sum_{j_1=1}^m(\hat{\pi}^1 A_{j_1,v^{[0]},t_1} f\circ u)\otimes(\Upsilon^2\circ p_{j_1}u)\}](y^{[n+1]})$.

Mention that in general $(\Upsilon^{n+1}f\circ u)(y^{[n+1]})$ may depend nontrivially on all components of the vector $y^{[n+1]}$ through several terms in Formula (7). Thus Formula (1) of this Lemma is proved by induction.

10. Corollary. *Let $u\in C^n(\mathbf{K}^s,\mathbf{K}^m)$, $u(\mathbf{K}^s)\subset U$ and $f\in C^n(U,Y)$, where U is an open subset in $\mathbf{K}^m$, $s,m\in\mathbf{N}$, Y is a $\mathbf{K}$-linear space, then*

(1) $\bar{\Phi}^n(f\circ u)(x^{(n)})=[\sum_{j_1=1}^m ... \sum_{j_n=1}^{m(n)}(B_{j_n,v^{(n-1)},t_n}...B_{j_1,v^{(0)},t_1} f\circ u)$
$(\bar{\Phi}^1\circ p_{j_n}\hat{S}_{j_{n-1}+1,v^{(n-2)}t_{n-1}}...\hat{S}_{j_1+1,v^{(0)}t_1}u^{n-1})(P_n\bar{\Phi}^1\circ p_{j_{n-1}}\hat{S}_{j_{n-2}+1,v_0^{(n-3)},t_{n-2}}...\hat{S}_{j_1+1,v^{(0)}t_1}u^{n-2})$
$...(P_n...P_2\bar{\Phi}^1\circ p_{j_1}u)+\sum_{j_1=1}^m ... \sum_{j_{n-1}=1}^{m(n-1)}(\hat{\pi}^1(B_{j_{n-1},v^{(n-2)},t_{n-1}}...B_{j_1,v^{(0)},t_1} f\circ u)[\sum_{\alpha=0}^{n-2}\hat{\pi}^{\otimes\alpha}\otimes$
$\bar{\Phi}\otimes\hat{P}^{\otimes(n-\alpha-2)}]$

$((\bar{\Phi}^1 \circ p_{j_{n-1}} \hat{S}_{j_{n-2}+1,v^{(n-3)}t_{n-2}} ... \hat{S}_{j_1+1,v^{(0)}t_1} u^{n-2}) \otimes ... \otimes (P_{n-1}...P_2 \bar{\Phi}^1 \circ p_{j_1} u))$
$+[\sum_{\alpha=0}^{n-2} \hat{\pi}^{\otimes\alpha} \otimes \bar{\Phi} \otimes \hat{P}^{\otimes(n-\alpha-2)}](\sum_{j_1=1}^m ... \sum_{j_{n-2}=1}^{m(n-2)} (\hat{\pi}^1 (B_{j_{n-2},v^{(n-3)},t_{n-2}} ... B_{j_1,v^{(0)},t_1} f \circ$
$u)) \otimes [\sum_{\alpha=0}^{n-3} \hat{\pi}^{\otimes\alpha} \otimes \bar{\Phi} \otimes \hat{P}^{\otimes(n-\alpha-3)}]((\bar{\Phi}^1 \circ p_{j_{n-2}} \hat{S}_{j_{n-3}+1,v^{(n-4)}t_{n-3}} ... \hat{S}_{j_1+1,v^{(0)}t_1} u^{n-3}) \otimes$
$... \otimes (P_{n-2}...P_2 \bar{\Phi}^1 \circ p_{j_1} u)) + ...$
$+[\sum_{\alpha=0}^{2} \hat{\pi}^{\otimes\alpha} \otimes \bar{\Phi} \otimes \hat{P}^{\otimes(2-\alpha)}]^{n-3} \{\sum_{j_1=1}^m \sum_{j_2=1}^{m(2)} (\hat{\pi}^1 B_{j_2,v^{(1)},t_2} B_{j_1,v^{(0)},t_1} f \circ u)(\bar{\Phi}^1 \otimes$
$\hat{P}^1 + \hat{\pi}^1 \otimes \bar{\Phi}^1)((\bar{\Phi}^1 \circ p_{j_2} \hat{S}_{j_1+1,v^{(0)}t_1} u) \otimes (P_2 \bar{\Phi}^1 \circ p_{j_1} u))\}$
$+(\bar{\Phi} \otimes \hat{P} + \hat{\pi} \otimes \bar{\Phi})^{n-2} \{\sum_{j_1=1}^m (\hat{\pi}^1 B_{j_1,v^{(0)},t_1} f \circ u) \otimes (\bar{\Phi}^2 \circ p_{j_1} u)\}](x^{(n)})$
and $f \circ u \in C^0((\mathbf{K}^s)^{(n)}, Y)$ *(see notation of Lemma 9), where* $B_{j,v,t} := (\hat{S}_{j+1,vt} \otimes t\bar{\Phi}^1 \circ p_j)^* \bar{\Phi}^1_j$*, where* $\bar{\Phi}^1$ *is taken for variables* (x,v,t) *or corresponding to them after actions of preceding operations as* $\bar{\Phi}^k$*,* $\bar{\Phi}^1_j f(x,v,t) := [f(x+e_j v_j t) - f(x)]/t$*,* $(B \otimes A)^* \bar{\Phi}^1 f_i \circ u^i(x,v,t) := \bar{\Phi}^1_j f_i(Bu^i, v, Au^i)$*,* $B : \mathbf{K}^{m(i)} \to \mathbf{K}^{m(i)}$*,* $A : \mathbf{K}^{m(i)} \to \mathbf{K}$*,* $m(i) = m + i - 1$*,* $j_i = 1, ..., m(i)$*,* $u^1 = u$*,* $u^2 := (u^1, t_1 \bar{\Phi}^1 \circ p_{j_1} u^1)$*,* $u^n := (u^{n-1}, t_{n-1} \bar{\Phi}^1 \circ p_{j_{n-1}} u^{n-1})$*,* $\hat{S}_* \bar{\Phi}^1 f(x) := \bar{\Phi}^1 f(\hat{S}_* x)$*.*

Proof. The restriction of operators of Lemma 9 on $W^{(n)}$ from Note 5 gives Formula (1) of this corollary, where $v^{(k)} \in (\mathbf{K}^s)^k \times \mathbf{K}^k$.

11. Lemma. *If* $a \ne 0$*,* $a \in \mathbf{K}$*,* U *is an open subset in* X*, where* X *and* Y *are topological vector spaces over* $\mathbf{K}$*,* $f \in C^1(U,Y)$*,* $T \in \mathbf{K}$*,* $T \ne 0$*, then*

(1) $\Upsilon^1 f(x, av, t/a) = a \Upsilon^1 f(x,v,t)$ *and*

(2) $\Upsilon^1 f(x,v,at) = a^{-1} \Upsilon^1 f(x,av,t)$ *and*

(3) $\Upsilon^1 f(x/T, v, t) = T^{-1} \Upsilon^1 f(x/T, v, t/T)$ *for each* $(x,v,t) \in U^{[1]}$ *and* $(x,v,at) \in U^{[1]}$ *and* $(x/T,v,t) \in U^{[1]}$ *respectively.*

Proof. We have identities: $\Upsilon^1 f(x,av,t/a) = [f(x+vta/a) - f(x)]/(t/a) = a[f(x+vt)-f(x)]/t = a\Upsilon^1 f(x,v,t)$, $\Upsilon^1 f(x,v,at) = [f(x+vta)-f(x)]/(at) = a^{-1}\Upsilon^1 f(x,av,t)$, for $g(x) := f(x/T)$ there is the equality $\Upsilon^1 g(x,v,t) = [g(x+vt) - g(x)]/t = [f((x+vt)/T) - f(x/T)]/t = T^{-1}[f(x/T + vt/T) - f(x/T)]/(t/T) = T^{-1}\Upsilon^1 f(x/T, v, t/T)$.

12. Lemma. *Let* $u : \mathbf{K} \to \mathbf{K}^b$ *be a polynomial function:*

(1) $u = \sum_{n=0}^m a_n x^n$,

where $a_n \in \mathbf{K}^b$ *are expansion coefficients,* $x \in \mathbf{K}$*,* $m \in \mathbf{N}$*, then*

(2) $\Upsilon^q u(x^{[q]}) = \sum_{n=1}^m a_n \sum_{k_1=1}^n \binom{n}{k_1} \{[\sum_{k_2=1}^{n-k_1} \binom{n-k_1}{k_2} ... \sum_{k_q=1}^{n-k_1-...-k_{q-1}} \binom{n-k_1-...-k_{q-1}}{k_q}$
$x^{n-k_1-...-k_q} ({}_1v_1^{[q-1]})^{k_q} t_q^{k_q-1} \mathcal{S}_{v^{[q-1]},t_q} ({}_1v_1^{[q-2]})^{k_{q-1}} (t_{q-1})^{k_{q-1}-1} ... \mathcal{S}_{v^{[1]},t_2} (v^{[0]})^{k_1} (t_1)^{k_1-1}]$
$+[x^{n-k_1-...-k_{q-1}} \sum_{k_2=1}^{k_1} \binom{n-k_1}{k_2} ... \sum_{k_{q-1}=1}^{n-k_1-...-k_{q-2}} \binom{n-k_1-...-k_{q-2}}{k_{q-1}} \sum_{k_q=1}^{k_{q-1}} \binom{k_{q-1}}{k_q}$
$({}_1v_1^{[q-2]})^{k_{q-1}-k_q} ({}_1v_2^{[q-1]})^{k_q} t_q^{k_q-1} \mathcal{S}_{v^{[q-1]},t_q} (t_{q-1})^{k_{q-1}-1}$

$\mathcal{S}_{v^{[q-2]},t_{q-1}}(\ _1v_1^{[q-3]})^{k_{q-2}}(t_{q-2})^{k_{q-2}-1}...\mathcal{S}_{v^{[1]},t_2}(v^{[0]})^{k_1}(t_1)^{k_1-1}] + ...$
$+[x^{n-k_1}(v^{[0]})^{k_1}\sum_{k_2=1}^{k_1-1}\binom{k_1-1}{k_2}...\sum_{k_q=1}^{k_{q-1}-1}\binom{k_{q-1}-1}{k_q}t_q^{k_q-1}$
$(v_3^{[q-1]})^{k_q}t_{q-1}^{k_{q-1}-k_q-1}...(v_3^{[2]})^{k_3}t_2^{k_2-k_3-1}(v_3^{[1]})^{k_2}t_1^{k_1-k_2-1}]\}$,
where $\mathcal{S}_{v^{[q-1]},t_q}\ _jx^{[q-1]} :=\ _jx^{[q-1]}+\ _jv^{[q-1]}t_q$ *for each* j, *where* $x^{[q]} = (\ _1x^{[q]},\ _2x^{[q]}, ...)$ *and this shift operator acts on all terms on the right of it in a product.*

Proof. In view of Lemma 3

(3) $\Upsilon^1u(x^{[1]}) = \sum_{n=0}^m a_n((x+v^{[0]}t_1)^n-x^n)/t_1 = \sum_{n=1}^m a_n\sum_{k_1=1}^n\binom{n}{k_1}x^{n-k_1}(v^{[0]})^{k_1}t_1^{k_1-1}$, where $\binom{n}{k}$ are binomial coefficients,

$\Upsilon^2u(x^{[2]}) = \sum_{n=1}^m a_n\sum_{k_1=1}^n\binom{n}{k_1}((x+v_1^{[1]}t_2)^{n-k_1}(v^{[0]}+v_2^{[1]}t_2)^{k_1}(t_1+v_3^{[1]}t_2)^{k_1-1}-x^{n-k_1}(v^{[0]})^{k_1}t_1^{k_1-1})/t_2$ in accordance with the notation of the proof of Lemma 4. Then

(4) $\Upsilon^2u(x^{[2]}) = \sum_{n=1}^m a_n\sum_{k_1=1}^n\binom{n}{k_1}\{[\sum_{k_2=1}^{n-k_1}\binom{n-k_1}{k_2}x^{n-k_1-k_2}(v_1^{[1]})^{k_2}t_2^{k_2-1}(v^{[0]}+v_2^{[1]}t_2)^{k_1}(t_1+v_3^{[1]}t_2)^{k_1-1}]+[x^{n-k_1}\sum_{k_2=1}^{k_1}\binom{k_1}{k_2}(v^{[0]})^{k_1-k_2}(v_2^{[1]})^{k_2}t_2^{k_2-1}(t_1+v_3^{[1]}t_2)^{k_1-1}]+[x^{n-k_1}(v^{[0]})^{k_1}\sum_{k_2=1}^{k_1-1}\binom{k_1-1}{k_2}t_1^{k_1-k_2-1}(v_3^{[1]})^{k_2}t_2^{k_2-1}]\}$.

Therefore, Formulas $(3,4)$ prove Formula (2) for $n=1$ and $n=2$. Let formula (2) be true for $n=1,...,q$, prove it for $n=q+1$. Applying to both sides of Equation (2) operator Υ^1 with the help of Formula 7(2) or 7(1) we get Formula (2) for $n=q+1$ also.

13. Corollary. *Let suppositions of Lemma 13 be satisfied, then* $|\Upsilon^qu(x^{[q]})|\le \max_{n=0}^m|a_n|$ *for each* $x^{[q]}\in\mathbf{K}^{[q]}$ *with* $|x^{[q]}|\le 1$.

Proof. The absolute value of each term on the right side of Formula 12(2) in the curled brackets is not greater than one, since binomial coefficients are integer numbers and their non-archimedean absolute value is not greater than one and each component of the vector $x_j^{[q]}\in\mathbf{K}$ has an absolute value not greater than one. Applying the non-archimedean inequality $|y+z|\le\max(|y|,|z|)$ for arbitrary $y,z\in\mathbf{K}$ we get the statement of this corollary.

14. Corollary. *Let* u *be a polynomial as in Lemma 13, then*

(1) $\bar{\Phi}^qu(x^{(q)}) = \sum_{n=q}^m a_n\sum_{k_1=1}^n\sum_{k_2=1}^{n-k_1}...\sum_{k_q=1}^{n-k_1-...-k_{q-1}}\binom{n}{k_1}\binom{n-k_1}{k_2}...\binom{n-k_1-...-k_{q-1}}{k_q}v_1^{k_1}...v_q^{k_q}t_1^{k_1-1}...t_q^{k_q-1}x^{n-k_1-...-k_q}$.

15. Lemma. *Let* $V_j\in\mathbf{R}$, $V_j>0$, *for each* $j\in\mathbf{N}$ *and* $\lim_{j\to\infty}V_j=0$. *Suppose also that* $g\in C^\infty(\mathbf{K}^l,\mathbf{K})$, *there exists* $R>0$ *such that* $g(x)=0$ *for each* $|x|>R$, *moreover,*

(1) $|\Upsilon^jg(x^{[j]})|\le C^{j+1}V_j^{-j}$

for each j *and* $|x^{[j]}|\le R$, *where* $C>0$ *is a constant. Put*

$u(x)=(a+\sum_{k_1,k_2=0}^m\sum_{i_1,i_2=1}^l \ _{i_1,i_2}b_{k_1,k_2}x_{i_1}^{k_1}x_{i_2}^{k_2})g(x/T)$
for each $x\in \mathbf{K}^l$, *where* $T\in\mathbf{K}$, $0<|T|\le 1$, a, $_{i_1,i_2}b_{k_1,k_2}\in\mathbf{K}$. *Then there exists a constant* $C_1>0$ *independent of* a, $_{i_1,i_2}b_{k_1,k_2}, j, x$ *and* T *such that*

(2) $\|\Upsilon^j u(x^{[j]})\|_{C^0(B(\mathbf{K}^{[j]},0,R),\mathbf{K})}$
$\le (\max_{i_1,i_2,k_1,k_2}(|a|,|\ _{i_1,i_2}b_{k_1,k_2}|))\max(1,R^m)|T|^{-j}C_1^{j+1}V_j^{-j}$.

Proof. Apply Lemmas 4 and 11. For this calculate by induction $\Upsilon^1(a+bx)(x,v,t)=bv$, $\Upsilon^2(a+bx)(x^{[2]})=bv_2^{[1]}$,..., $\Upsilon^j(a+bx)(x^{[j]})=bv_2^{[j-1]}$ for each $j\ge 3$. Therefore, $\|\Upsilon^j(a+bx)\|_{C^0(B(\mathbf{K}^{[j]},0,R),\mathbf{K})}\le\max(|a|,|b|)R$ for each $j\ge 0$. In general apply Formula 12(2) and Corollary 13. Then by induction from Formula 11(3) it follows, that $\|\Upsilon^j g(x/T)\|_{C^0(B(\mathbf{K}^{[j]},0,R),\mathbf{K})}=|T|^{-j}\|\Upsilon^j g(x)\|_{C^0(B(\mathbf{K}^{[j]},0,R),\mathbf{K})}$ for each $j\ge 0$, where $\Upsilon^0 g=g$. Therefore, from Formula 4(1) and the ultrametric inequality we have

$\|\Upsilon^j u(x^{[j]})\|_{C^0(B(\mathbf{K}^{[j]},0,R),\mathbf{K})}\le$
$(\max_{i_1,i_2,k_1,k_2}(|a|,|\ _{i_1,i_2}b_{k_1,k_2}|))\max(1,R^m)\max_{k=0}^j|T|^{-k}\|\Upsilon^k g(x)\|_{C^0(B(\mathbf{K}^{[k]},0,R),\mathbf{K})}\le$
$(\max_{i_1,i_2,k_1,k_2}(|a|,|\ _{i_1,i_2}b_{k_1,k_2}|))\max(1,R^m)\max_{k=0}^j|T|^{-k}C^{k+1}V_k^{-k}$,
since $g(x)=0$ for $|x|>R$ and choosing $C_1>0$ such that $\infty>C_1\ge\sup_{j=0}^\infty[\sup_{k=0}^j(C^{k+1}V_k^{-k}V_j^j|T|^{j-k})^{1/(j+1)}]$ we get the statement of this Lemma.

16. Lemma. *If* U *is an open subset in* $\mathbf{K}^b$, $f:U\to\mathbf{K}$ *is a marked function, then a space* Y_n *of functions* $\{\Upsilon^n f(x^{[n]}): v^{[0]},\ _lv_j^{[k]}\in\{0,1\}; j=1,2; l=1,2,...; k=1,...,n-1\}$ *is finite dimensional over* $\mathbf{K}$ *whenever it exists such that* $dim_{\mathbf{K}}Y_n\le(2^{m(n-1)}-1)dim_{\mathbf{K}}Y_{n-1}$, $n\in\mathbf{N}$, $m(n)=2m(n-1)+1$ *for* $n\in\mathbf{N}$, $m(0)=b$.

Proof. We have the recurrence relation for a number of variables belonging to $\mathbf{K}$, $m(n)=2m(n-1)+1$ for each $n\in\mathbf{N}$ corresponds to $\Upsilon^n f(x^{[n]})$, $m(0)=b$ corresponds to $f(x)$. For $n=1$ we have

$\Upsilon^1 f(x,v,t)=(f(x+vt)-f(x))/t=[f(x+vt)-f(x+(v-\ _bve_b)t)]/t+[f(x+(v-\ _bve_b)t)-f(x+(v-\ _be_b-\ _{b-1}e_{b-1})t)]/t+...+[f(x+\ _1ve_1t)-f(x)]/t$, where $v=(\ _1v,...,\ _bv)$, $_lv\in\mathbf{K}$ for each $l=1,...,b$. We have that $_lv\in\{0,1\}$ may take only two values and the amount of such nonzero vectors v is equal to 2^b-1. Thus the family $\{\Upsilon^1 f(x+(v-\ _bve_b-...-\ _kve_k),\ _kve_k,t):\ _lv\in\{0,1\}, l=1,...,b\}$ of functions by $(x,t)\in\mathbf{K}^{b+1}$ spans over $\mathbf{K}$ the space $\{\Upsilon^1 f(x,v,t):\ _lv\in\{0,1\}, l=1,...,b\}$. Its dimension over $\mathbf{K}$ for a given f is not greater, than 2^b-1.

Let the statement of this lemma be true for $n-1\ge 1$. Then apply the operator Υ^1 to $\Upsilon^{n-1}f(x^{[n-1]})$. Replacing in the proof above f on $f^{[n-1]}$ we get the statement of this lemma for n also, since $\Upsilon^n f(x^{[n]})=$

$\Upsilon^1(\Upsilon^{n-1}f(x^{[n-1]}))((x^{[n-1]})^{[1]})$ and $x^{[n]}=(x^{[n-1]})^{[1]}$ considering $f^{[n]}(x^{[n]})$ by free variables $(x,t_1,...,t_n)$.

17. Corollary. *For each $n\in \mathbf{N}$ and each $b\in \mathbf{N}$ and a marked function $f:U\to \mathbf{K}$, where U is open in $\mathbf{K}^b$ there exists a finite system Λ_n of vectors $0\ne (y,v)$, $y,v\in \mathbf{K}^{m(n-1)}$ such that*

(1) $\sum_{(y,v)\in \Lambda_n} C_{(y,v)}\Upsilon^n f(x^{[n-1]}+y,v,t_n)=0$

is identically equal to zero as the function of $(x,t_1,...,t_n)$, where $(x^{[n-1]}+y,v,t_n)\in U^{[n]}$, $x^{[n-1]}\in U^{[n-1]}$, $v^{[0]}$ and ${}_lv_j^{[k]}\in\{0,1\}$ for each j,k,l, y may depend on the parameters $t_1,...,t_n$ polynomially, $0\ne C_{(y,v)}\in \mathbf{K}$ are constants for each (y,v).

Proof. Take $card(\Lambda_n)>dim_{\mathbf{K}}Y_n$ and ${}_lv_j^{[k]}\in\{0,1\}$ for each $l=1,...,m(k-1)$, $j=1,2$ and $k=0,...,n-1$ such that $(x,v,t_1)\in U^{[1]}$. Then

$\Upsilon^1 f(x,v,t_1)=\Upsilon^1 f(x+(v-{}_bve_b)t_1,{}_bve_b,t_1)+\Upsilon^1 f(x+(v-{}_bve_b-{}_{b-1}ve_{b-1})t,{}_{b-1}ve_{b-1},t_1)+...+\Upsilon^1 f(x,{}_1ve_1,t_1)$,

hence Υ^f on vectors $\{(x,v,t_1);(x+(v-{}_bve_b)t_1,{}_bve_b,t_1);(x+(v-{}_bve_b-{}_{b-1}ve_{b-1})t_1,{}_{b-1}ve_{b-1},t_1);...;(x,{}_1ve_1,t_1)\}$ is $\mathbf{K}$-linearly dependent system of functions by (x,t_1), where ${}_lv\in\{0,1\}$, $l=1,...,b$, $C_{(y,v)}\ne 0$. Let the statement be proved for $n-1$, then prove it for n. Apply to both sides of equation

$\sum_{(y^{n-1},v^{n-1})\in\Lambda_{n-1}} C_{(y^{n-1},v^{n-1})}\Upsilon^{n-1}f(x^{[n-2]}+y^{n-1},v^{n-1},t_{n-1})=0$

operator Υ^1, which is $\mathbf{K}$-linear, consequently,

$\sum_{(y^1,v^1)\in\Lambda_1} C_{(y^1,v^1)}\Upsilon^1(\sum_{(y^{n-1},v^{n-1})\in\Lambda_{n-1}}\Upsilon^{n-1}f)((x^{[n-2]}+y^{n-1},v^{n-1},t_{n-1})+y^1,v^1,t_n)=0$,

where Λ_1, y^1 and v^1 already correspond to $\Upsilon^{n-1}f(x^{[n-1]})$ instead of $f(x)$, we get Formula (1) with $C_{(y^n,v^n)}=C_{(y^1,v^1)}C_{(y^{n-1},v^{n-1})}\ne 0$ and with $\Upsilon^n f(x^{[n-2]}+y^{n-1},v^{n-1},t_{n-1})+y^1,v^1,t_n)=\Upsilon^n f(x^{[n-1]}+y^n,v^n,t_n)$.

18. Corollary. *If U is an open subset in $\mathbf{K}^b$, $f:U\to\mathbf{K}$ is a marked function, then a space X_n of functions $\{\bar\Phi^n f(x^{(n)}):{}_lv_j\in\{0,1\},l=1,...,b;j=1,...,n\}$ is finite dimensional over $\mathbf{K}$ whenever it exists such that $dim_{\mathbf{K}}X_n\le(2^b-1)^n$, $n\in\mathbf{N}$. Moreover, there exists a finite system Λ_n of vectors $0\ne(y,v)$, $y\in\mathbf{K}^b$, $v\in(\mathbf{K}^b)^n$ such that*

(1) $\sum_{(y,v)\in\Lambda_n} C_{(y,v)}\bar\Phi^n f(x+y,v,t_1,...,t_n)=0$

is identically equal to zero as the function of $(x,t_1,...,t_n)$, where $(x+y,v,t_1,...,t_n)\in U^{(n)}$, $x^{(n-1)}\in U^{(n-1)}$, $v^{(0)}$ and ${}_lv_j\in\{0,1\}$ for each $l=1,...,b$, $j=1,...,n$, y may depend on the parameters $t_1,...,t_n$ linearly, $0\ne C_{(y,v)}\in\mathbf{K}$ are constants

for each (y,v).

Proof. Restrict in the preceding formulas $\Upsilon^n f(x^{[n]})$ on $W^{(n)}$ and from Lemma 25 and Corollary 26 we get the statement of this corollary.

19. Lemma. *Let U be an open subset in $\mathbf{K}^m$, Y be a $\mathbf{K}$-linear space. If $char(\mathbf{K})=0$, then either $f\in C^{[n]}(U,Y)\cap C^{n+1}(U,Y)$ or $f\in C^{[n]}_b(U,Y)\cap C^{n+1}_b(U,Y)$ if and only if either $f\in C^{[n+1]}(U,Y)$ or $f\in C^{[n+1]}_b(U,Y)$. If $char(\mathbf{K})>0$, then $C^{[n]}(U,Y)\subset C^n(U,Y)$ and $C^{[n]}_b(U,Y)\subset C^n_b(U,Y)$.*

Proof. If $f\in C^{[n+1]}(U,Y)$ or $f\in C^{[n+1]}_b(U,Y)$, then the restriction $\Upsilon^{n+1}f|_{W^{(n+1)}}=\bar{\Phi}^{n+1}f$ is continuous or uniformly continuous on $V^{(n+1)}$ correspondingly, consequently, $f\in C^{n+1}(U,Y)$ or $f\in C^{n+1}_b(U,Y)$ respectively. Since $C^{[n]}(U,Y)\subset C^{[n+1]}(U,Y)$ or $C^{[n]}_b(U,Y)\subset C^{[n+1]}_b(U,Y)$, then $f\in C^{[n]}(U,Y)\cap C^n(U,Y)$ or $f\in C^{[n]}_b(U,Y)\cap C^n_b(U,Y)$ correspondingly (see also Note 5).

Let now $f\in C^{[n]}(U,Y)\cap C^{n+1}(U,Y)$ or $f\in C^{[n]}_b(U,Y)\cap C^{n+1}_b(U,Y)$. For $n=0$ the statement of this lemma follows from Lemma 2. Suppose that the statement of this lemma is true for $k=1,...,n$, then prove it for $k=n+1$. In view of Lemma 9 we have, that $f^{[n+1]}(x^{[n]})$ has the expression through the finite sum of terms $(\hat{\Phi}^{n+1}f\circ u^{n+1})h_\beta$ up to minor terms $(\Upsilon^i f)h_\beta$ with $i\le n$, where $u^{n+1}\in C^\infty_b$ and $h_\beta\in C^\infty_b$ are functions associated with $\mathbf{K}$-linear and polynomial shift operators and their compositions independent of f. We can write this in more details by induction. Now consider the composite function:

(1) $g(x^{[q]}):=(\bar{\Phi}^q f)(u(x;\alpha);e_{j_1},...,e_{j_q};a_1\bar{\Phi}^{n(1)}u_{k_1}(x;\ {_1e};b_1),...,$
$a_s\bar{\Phi}^{n(s)}u_{k_s}(x;\ {_se};b_s))\bar{\Phi}^{m(1)}w_1(x;\ {_1\xi};c_1)...\bar{\Phi}^{m(r)}w_r(x;\ {_r\xi};c_r)$

appearing from the decomposition of $f^{[q]}$, where $a_1,...,a_s$ are polynomials of $t_1,...,t_q$ and ${_lv^{[k]}_j}$, $k=0,...,q$, $l=1,2,...$, $j=1,2,3$ for $k>0$, $b_l\subset\{t_1,...,t_q\}$, ${_r\xi}=({_re_{i_1}},...,{_re_{i_{m(r)}}})$, ${_se}=({_se_{j_1}},...,{_se_{j_{n(s)}}})$; $k_j,s,r,n(s),m(r)\in\mathbf{N}$; α is a parameter, $w_1,...,w_r$ are polynomials of $x,t_1,...,t_q,\ {_lv^{[k]}_j}$;
here $\alpha,u,a_1,...,a_s,w_1,...,w_r,e_i,\ {_j\xi},b_j$ are independent of f. The set of variables $(x;t_1,...,t_q;\ {_lv^{[k]}_j}:l=1,2,...;k=0,...,q;j=1,2,3\}$ is in the bijective correspondence with the vector $x^{[q]}$. Then act on this function g by the operator Υ^1 at the vector $x^{[q+1]}=(x^{[q]},v^{[q]},t_{q+1})$ such that $\Upsilon^1g(x^{[q+1]})=[g(x^{[q]}+v^{[q]}t_{q+1})-g(x^{[q]})]/t_{q+1}$. For the calculation of Υ^1g apply Formulas 7(2) and 9(2) to g and $(\bar{\Phi}^q f)$ by all variables of functions in this composition and product. It is nonlinear by $\bar{\Phi}^q f$. As the result Υ^1g is the $\mathbf{K}$-linear combination of functions of the same type (1) with $q+1$ instead of q and in

general new functions in the composition and product after actions on them operators Υ^1, P_k, $\hat{\pi}$ and S. Shift operators $\hat{S}$ over $\mathbf{K}$ are infinite differentiable and invertible such that for $\mathbf{K}$ with $char(\mathbf{K})=0$ we have $\sum_{i=1}^k \hat{S} = k\hat{S} \ne 0$ and $\sum_{i=1}^k t\bar{\Phi}^1 id(x;v;t) = kt\bar{\Phi}^1 id(x;v;t) \ne 0$ for all $k \in \mathbf{N} = \{1,2,3,...\}$ and each $t \ne 0$ and $v \ne 0$.

20. Corollary. *If $char(\mathbf{K})=0$, then a function f belongs to $C^{[n+1]}(U,Y)$ or $C^{[n+1]}_b(U,Y)$ if and only if f belongs to $C^{n+1}(U,Y)$ or $C^{n+1}_b(U,Y)$ respectively, moreover, there exists a constant $0 < C_1 < \infty$ independent of f such that $\|f\|_n \le \|f\|_{[n]} \le C_1\|f\|_n$ for each $f \in C^{n+1}_b(U,Y)$, where $V^{[k]} := \{x^{[k]} \in U^{[k]} : |v_1^{[q]}| = 1; |\,_lv_2^{[q]}t_{q+1}| \le 1, |v_q^{[3]}| \le 1 \quad \forall l,q\}$ or $V^{(k)} := \{x^{(k)} \in U^{(k)} : |v_j| = 1 \quad \forall j\}$ with norms either*

$\|f\|_{[n]} := \sup_{k=0,...,n; x^{[k]} \in V^{[k]}} |f^{[k]}(x^{[k]})|$ *or*

$\|f\|_n := \sup_{k=0,...,n; x^{(k)} \in V^{(k)}} |\bar{\Phi}^k f(x^{(k)})|$.

Proof. Apply Lemma 19 by induction for $k=1,...,n$ and use Lemma 2. If g is a bounded continuous function $g : \mathbf{K}^m \to \mathbf{K}$, then $[g(x+vt)-g(x)]/t$ is a bounded continuous function by $(x,v,t) \in \mathbf{K}^m \times \mathbf{K}^m \times (\mathbf{K} \setminus B(\mathbf{K},0,\delta))$, where $\delta > 0$ is a constant. If $L(X,Y)$ is the space of all bounded $\mathbf{K}$ linear operators $T : X \to Y$ from a normed space X into a normed space Y over $\mathbf{K}$, then operator norms $\|T\|_1 := \sup_{0 \ne x \in X} \|Tx\|_Y/\|x\|_X$, $\|T\|_2 := \sup_{0<|x|\le 1, x \in X} \|Tx\|_Y/\|x\|_X$ and $\|T\|_3 := \sup_{|x|=1, x\in X} \|Tx\|_Y/\|x\|_X$ are equivalent [18]. In view of Lemma 2 each operator $\bar{\Phi}^j f(x;v_1,...,v_j;0,...,0)$ is j multi-linear over $\mathbf{K}$ by vectors $v_1,...,v_j \in X$. Therefore, the definition of the C^n norm given above is worthwhile. If $x^{[k]} \in V^{[k]}$, then $|v_1^{[q]}| = 1$ and $|\,_lv_1^{[q-1]} + \,_lv_2^{[q]}t_{q+1}| \le 1$ for each l,q. The inequality $\|f\|_n \le \|f\|_{[n]}$ follows from $\bar{\Phi}^n f = f^{[n]}_{W^{(n)}}$. The second inequality $\|f\|_{[n]} \le C_1\|f\|_n$ follows from the decomposition of $f^{[q]}$ as a finite $\mathbf{K}$-linear combination of terms having the form 19(1) for each $q=1,...,n$ and since norms of all terms are bounded and expansion coefficients are independent of f, where $f^{[0]} = f$.

21. Lemma. *Let U be an open subset in $\mathbf{K}^b$, $b \in \mathbf{N}$, let also $f : U \to Y$ be a function with values in a topological vector space Y over $\mathbf{K}$. Then $f \in C^{[n]}(U,Y)$ or $f \in C^{[n]}_b(U,Y)$ or $f \in C^n(U,Y)$ or $f \in C^n_b(U,Y)$ if and only if $\Upsilon^k f(x^{[k]}) \in C^0(U^{[k]}_{j(0),j(1),...,j(k)}, Y)$ or $\Upsilon^k f(x^{[k]}) \in C^0_b(V^{[k]}_{j(0),j(1),...,j(k)}, Y)$ or $\bar{\Phi}^k f(x;e_{j(1)},...,e_{j(k)};t_1,...,t_k) \in C^0(U^{(k)}_{j(1),...,j(k)}, Y)$ or $\bar{\Phi}^k f(x;e_{j(1)},...,e_{j(k)};t_1,...,t_k) \in C^0_b(V^{(k)}_{j(1),...,j(k)}, Y)$ for each $k=0,1,...,n$ and for each $j(i) \in \{1,...,m_i\}$, $v^{[i]} = e_{j(i)} \in (\mathbf{K}^b)^{[i]}$, $m_i = dim_{\mathbf{K}}(\mathbf{K}^b)^{[i]}$, $i =$*

$0,1,...,k$, $x^{[i+1]} = (x^{[i]}, v^{[i]}, t_{i+1})$ *or respectively each* $j(1),...,j(k) \in \{1,2,...,b\}$ *with* $e_j = (0,...,0,1,0,...,0) \in \mathbf{K}^b$ *is the vector with* 1 *on the* j*-th place, where* $U^{[k]}_{j(0),...,j(l)} := \{x^{[k]} \in U^{[k]} : v^{[i]} = e_{j(i)}, i = 0,...,l\}$, $V^{[k]}_{j(0),...,j(l)} = V^{[k]} \cap U^{[k]}_{j(0),...,j(l)}$, $U^{(k)}_{j(1),...,j(l)} := \{x^{(k)} \in U^{(k)} : v_1 = e_{j(1)},...,v_l = e_{j(l)}\}$, $V^{(k)}_{j(1),...,j(l)} = V^{(k)} \cap U^{(k)}_{j(1),...,j(l)}$. *Moreover, if each* $\Upsilon^k f(z^{[k]})|_{V^{[k]}_{j(0),...,j(k)}}$ *or* $\bar{\Phi}^k f(z; e_{j(1)},...,e_{j(n)}; t_1,...,t_n)$ *is locally bounded, then* $\Upsilon^k f(z^{[k]})$ *or* $\bar{\Phi}^k f(z^{(k)})$ *is locally bounded respectively.*

Proof. If $n=1$, then

(1) $\bar{\Phi}^1 f(x; v_1; t_1) = \bar{\Phi}^1 f({}_1x, {}_2x + {}_2v_1t_1, ..., {}_bx + {}_bv_1t_1; e_1; {}_1v_1t_1)\, {}_1v_1 + \bar{\Phi}^1 f({}_1x, {}_2x, {}_3x + {}_3v_1t_1, ..., {}_bx + {}_bv_1t_1; e_2; {}_2v_1t_1)\, {}_2v_1 + ...$
$+ \bar{\Phi}^1 f({}_1x, ..., {}_bx; e_b; {}_bv_1t_1)\, {}_bv_1$,

hence $\bar{\Phi}^1 f(x; v_1; t_1) \in C^0(U^{(1)}, Y)$ or $\bar{\Phi}^1 f(x; v_1; t_1) \in C^0_b(V^{(1)}, Y)$ if and only if $\bar{\Phi}^1 f(x; e_{j(1)}; t_1) \in C^0(U^{(1)}_{j(1)}, Y)$ or $\bar{\Phi}^1 f(x; e_{j(1)}; t_1) \in C^0_b(V^{(1)}_{j(1)}, Y)$ for each $j(1) \in \{1,...,b\}$, where $x = ({}_1x, ..., {}_bx)$, ${}_jx \in \mathbf{K}$ for each j, $\Upsilon^1 f = \bar{\Phi}^1 f$. In accordance with Formula 10(1) or 9(1) we have the expression of $\bar{\Phi}^k f(x; v_1,...,v_k; t_1,...,t_k)$ or $\Upsilon^k f(x^{[k]})$ throughout the sum of terms containing $\bar{\Phi}^k f(x; e_{j(1)},...,e_{j(k)}; t_1,...,t_k)$ or $\Upsilon^k f(z^{[k]})|_{V^{[k]}_{j(0),...,j(k)}}$ with multipliers belonging to $C^\infty_b(U,Y)$ or $C^{[\infty]}_b(U,Y)$ putting in Formula 10(1) or 9(1) $u = id : \mathbf{K}^b \to \mathbf{K}^b$, $id(x) = x$ for each x, $s = m = b$. From this the second assertion follows.

Suppose that the first statement of this lemma is proved for all $k = 0,1,...,n-1$. Then apply the operator $\bar{\Phi}^1$ to each $\bar{\Phi}^{n-1} f(x; e_{j(1)},...,e_{j(n-1)}; t_1,...,t_{n-1})$ and in accordance with Formula (1) with $\bar{\Phi}^{n-1} f$ or Υ^1 to each $\Upsilon^{n-1} f(x^{[n-1]})$ with $x^{[n-1]} \in U^{[n-1]}_{j(0),...,j(n-1)}$ instead of f we get the same conclusion. Thus $\bar{\Phi}^n f(x; e_{j(1)},...,e_{j(n-1)}, v_n; t_1,...,t_{n-1}, t_n)$ belongs to C^0 or C^0_b by its variables belonging to $U^{(n)}_{j(1),...,j(n-1)}$ or to $V^{(n)}_{j(1),...,j(n-1)}$ or $\Upsilon^n f(x^{[n]})$ belongs to C^0 or C^0_b by $x^{[n]} \in U^{[n]}_{j(0),...,j(n-1)}$ or $x^{[n]} \in V^{[n]}_{j(0),...,j(n-1)}$ respectively if and only if $\bar{\Phi}^{n-1} f(x; e_{j(1)},...,e_{j(n-1)}, e_{j(n)}; t_1,...,t_{n-1}, t_n)$ belongs to $C^0(U^{(n)}_{j(1),...,j(n)}, Y)$ or $C^0_b(V^{(n)}_{j(1),...,j(n)}, Y)$ or $\Upsilon^n f(x^{[n]})|_{U^{[n]}_{j(0),...,j(n)}} \in C^{[n]}(U^{[n]}_{j(0),...,j(n)}, Y)$ or $\Upsilon^n f(x^{[n]})|_{V^{[n]}_{j(0),...,j(n)}} \in C^{[n]}_b(V^{[n]}_{j(0),...,j(n)}, Y)$ respectively for each $j(n)$, where $j(0),...,j(n)$ are arbitrary. Together with the induction hypothesis this finishes the proof of this lemma.

22. Lemma. *Suppose that* U^k *is open in* $\mathbf{K}^k$ *for each* $2 \le k \le m$ *with* $domain(f_k) = U^k$ *and from* $f_k \circ u \in C^{[n]}(\mathbf{K}^{k-1}, Y)$ *or* $C^{[n]}_b(\mathbf{K}^{k-1}, Y)$ *or*

$C^n(\mathbf{K}^{k-1},Y)$ or $C^n_b(\mathbf{K}^{k-1},Y)$ for each $u \in C^{[\infty]}(\mathbf{K}^{k-1},\mathbf{K}^k)$ or $C^{[\infty]}_b(\mathbf{K}^{k-1},\mathbf{K}^k)$ or $C^\infty(\mathbf{K}^{k-1},\mathbf{K}^k)$ or $C^\infty_b(\mathbf{K}^{k-1},\mathbf{K}^k)$ with $image(u) \subset U^k$ it follows that $f_k \in C^{[n]}(U^k,Y)$ or $C^{[n]}_b(U^k,Y)$ or $C^n(U^k,Y)$ or $C^n_b(U^k,Y)$ respectively. Then for $domain(f) = U$ open in $\mathbf{K}^m$ from $f \circ u \in C^{[n]}(\mathbf{K},Y)$ or $C^{[n]}_b(\mathbf{K},Y)$ or $C^n(\mathbf{K},Y)$ or $C^n_b(\mathbf{K},Y)$ for each $u \in C^{[\infty]}(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K},\mathbf{K}^m)$ or $C^\infty(\mathbf{K},\mathbf{K}^m)$ or $C^\infty_b(\mathbf{K},\mathbf{K}^m)$ with $image(u) \subset U$ it follows, that $f \in C^{[n]}(\mathbf{K},Y)$ or $C^{[n]}_b(\mathbf{K},Y)$ or $C^n(\mathbf{K},Y)$ or $C^n_b(\mathbf{K},Y)$ respectively.

Proof. Write $f \circ u$ in the form $f \circ u = f \circ u_{m-1} \circ u_{m-2} \circ ... \circ u_1$, where $u_j : \mathbf{K}^j \to \mathbf{K}^{j+1}$ for each j and $u : \mathbf{K} \to \mathbf{K}^m$ of corresponding classes of smoothness. Applying supposition of this lemma for $k = m, m-1, ..., 2$ we get that $f \circ u_{m-1} \circ ... \circ u_j \in C^{[n]}(\mathbf{K}^j,Y)$ provides $f \in C^{[n]}(U,Y)$ or also for others classes of smoothness correspondingly for each $j = m-1, m-2, ..., 1$.

23. Lemma. *Let $f : U \to \mathbf{K}^l$, where U is open in $\mathbf{K}^m$. Then $f \in C^{[n]}(U,\mathbf{K}^l)$ if and only if each $\Upsilon^n f(x^{[n]})_{U^{[n]}_{j(0),...,j(n)}}$ is continuous for $v^{[k]}_3 = 0$ for each $k = 1, ..., n-1$.*

Proof. In view of Lemma 21 it remains to prove, that continuity of each $\Upsilon^n f(x^{[n]})_{U^{[n]}_{j(0),...,j(n)}}$ is equivalent to the continuity of this family under the condition $v^{[k]}_3 = 0$ for each $k = 1, ..., n-1$. Prove this by induction. We already have that $v^{[k]}_3 \in \{0,1\}$ for each $k = 0, ..., n-1$. Denote by $\hat{S}_{n,t_n}$ the shift operator $\hat{S}_{n,t_n} g(t_{n-1},\beta) := g(t_{n-1}+t_n)$, where β denotes the family of all other variables of a function g. Then $\Upsilon^n f(x^{[n]}) = [\hat{S}_{n,t_n}\Upsilon^{n-1} f(x^{[n-1]}+w^{[n-1]}t_n) - \Upsilon^{n-1} f(x^{[n-1]})]/t_n = [(\hat{S}_{n,t_n}-I)\Upsilon^{n-1} f(x^{[n-1]}+ w^{[n-1]}t_n)]/t_n + \Upsilon^n f(x^{[n]})|_{v^{[n-1]}_3=0}$,
where $w^{[n-1]}$ differs from $v^{[n-1]}$ by $v^{[n-1]}_3$ such that in $w^{[n-1]}$ it is zero and in $v^{[n-1]}$ it is one while all others their components coincide such that $x^{[n]} = (x^{[n-1]}, v^{[n-1]}, t_n)$, $x^{[n]}|_{v^{[n-1]}_3=0} = (x^{[n-1]}, w^{[n-1]}, t_n)$. Since $[(\hat{S}_{n,t_n}-I)\Upsilon^{n-1} f(x^{[n-1]}+ w^{[n-1]}t_n)]/t_n = [\Upsilon^{n-1} f((x^{[n-2]}, v^{[n-2]}, t_{n-1}+t_n) + w^{[n-1]}t_n) - \Upsilon^{n-1} f(x^{n-1} + w^{[n-1]}t_n)]/t_n$, where $x^{[0]} = x$, $x^{[n-1]} = (x^{[n-2]}, v^{[n-2]}, t_{n-1})$, $n \ge 2$ and $k \ge 1$, then $[(\hat{S}_{n,t_n} - I)\Upsilon^{n-1} f(x^{[n-1]} + w^{[n-1]}t_n)]/t_n = \Upsilon^1_{s(n-1)}\Upsilon^{n-1} f(x^{[n-1]} + v^{[n-1]}t_n)(t_{n-1}+t_n-t_{n-1})/t_n$ is continuous, where $s(n-1)$ corresponds to the partial difference quotients by the variable t_{n-1}. Then by induction get that $(\hat{S}_{k,t_k} - I)/t_k = \Upsilon^1_{s(k)}$ for each $k = n-1, ..., 1$ which leads to the assertion of this lemma.

24. Lemma. *Suppose that $f \in C^n(U,Y)$ or $f \in C^{[n]}(U,Y)$, where U is*

open in $\mathbf{K}^m$, then each $\bar\Phi^n f(x^{(n)})$ has the symmetry by transposition of pairs (v_j, t_j) characterized by the Young tableaux consisting of one row of length n, each $\Upsilon^n f(x^{[n]})|_{\{U^{[n]}: v_3^{[k]}=0, k=1,...,n\}}$ is characterized by the Young tableaux consisting of 2^{n-1} rows, where the first row of length n contains numbers $1,...,n$, the second row of length $n-1$ contains numbers $2,...,n$, the third and the fourth rows have lengths $n-2$ and contain numbers $3,...,n$ and so on, where the number of rows of equal lengths $n-k$ is 2^{k-1} for $1 \le k < n-1$. Moreover, if $t_{i_1} = 0$,...,$t_{i_l} = 0$ as arguments of $\Upsilon^n f$, then its symmetry becomes higher with the amount of rows 2^{n-l} instead of 2^{n-1}.

Proof. The function $\bar\Phi^n f(x; v_1, ..., v_n; t_1, ..., t_n)$ is symmetric relative to transpositions $(v_i, t_i) \mapsto (v_j, t_j)$, since $[(f(x+v_it_i+v_jt_j) - f(x+v_jt_j))/t_i - (f(x+v_it_i) - f(x))/t_i]/t_j = [(f(x+v_it_i+v_jt_j) - f(x+v_it_i))/t_j - (f(x+v_jt_j) - f(x))/t_j]/t_i$ for each $i \ne j$ and so on by induction.

When $v_3^{[k]} = 0$ for $1 \le k \le n-1$ and $n \ge 2$ we have

(1) $\Upsilon^{k+1} f(x^{[k+1]}) = \{[\Upsilon^{k-1} f(x^{[k-1]} + (v^{[k-1]} + v_2^{[k]} t_{k+1}) t_k + v_1^{[k]} t_{k+1}) - \Upsilon^{k-1} f(x^{[k-1]} + v_1^{[k]} t_{k+1})]/t_k - [\Upsilon^{k-1} f(x^{[k-1]} + v^{[k-1]} t_k) - \Upsilon^{k-1} f(x^{[k-1]})]/t_k\}/t_{k+1} = \{[\Upsilon^{k-1} f(x^{[k-1]} + (v^{[k-1]} + v_2^{[k]} t_{k+1}) t_k + v_1^{[k]} t_{k+1}) - \Upsilon^{k-1} f(x^{[k-1]} + v^{[k-1]} t_k + v_1^{[k]} t_{k+1})]/t_k + [\Upsilon^{k-1} f(x^{[k-1]} + v^{[k-1]} t_k + v_1^{[k]} t_{k+1}) - \Upsilon^{k-1} f(x^{[k-1]} + v_1^{[k]} t_{k+1})]/t_k - [\Upsilon^{k-1} f(x^{[k-1]} + v^{[k-1]} t_k) - \Upsilon^{k-1} f(x^{[k-1]})]/t_k\}/t_{k+1}$

and this expression is symmetric relative to transpositions $(v^{[k-1]}, t_k) \mapsto (v_1^{[k]}, t_{k+1})$. Therefore, exclude $v_3^{[k]} = 0$ from the consideration such that $v^{[0]} := v^{[0],1}$, $v^{[1]} = (v^{[1],1}, v^{[1],2}, 0)$, where $v^{[0],1}, v^{[1],1}, v^{[1],2} \in \mathbf{K}^m$. Then by induction define vectors $v^{[k],i} \in \mathbf{K}^m$ such that $x^{[k]} + v^{[k]} t_{k+1} = (x^{[k-1]} + v_1^{[k]} t_{k+1}, v^{[k-1]} + v_2^{[k]} t_{k+1}, t_k + v_3^{[k]} t_{k+1})$ with $v_3^{[k]} = 0$ and to this corresponds $v^{[k-1],i} + v^{[k],i+2^{k-1}} t_{k+1}$ such that $v^{[k]}$ is completely characterized by $(v^{[k],i} : i = 1, ..., 2^k)$, where $k \ge 1$. Therefore, by induction $\Upsilon^n f$ is symmetric relative to transpositions $(v^{[k-1],i}, t_k) \mapsto (v^{[k],i}, t_{k+1})$ for each $1 \le i \le 2^{k-1}$, $1 \le k \le n-1$. To $v^{[k],1}$ pose the first row of length n with numbers $1, ..., n$ in boxes from left to right, $k = 0, 1, ..., n-1$. To vectors $v^{[k],i}$ with $i = 2^{k-1}+1, ..., 2^k$ and $k \ge 1$ pose rows in the Young tableaux with such numbers in squares from left to right beginning with $k+1$ and ending with n in each such i-th row.

If $t_{i_1} = 0$,...,$t_{i_l} = 0$ as arguments of $\Upsilon^n f$, then the symmetry of $\Upsilon^n f$ up to notation corresponds to $v^{[i_s-2]} + v_2^{[i_s-1]} t_{i_s} = v^{[i_s-2]}$ and $\Upsilon^n f$ is characterised by less amount of vectors $v^{[k],i}$, since $\Upsilon^l_{t_{i_1},...,t_{i_l}} f = \bar\Phi^l_{t_{i_1},...,t_{i_l}} f$ such that instead of $(v^{[k-1],j} : j = 1, ..., 2^{k-1})$ it is sufficient to take $j = 1, ..., 2^{k-2}$ for $k = i_2$ for

$k \geq 2$ and so on excluding excessive vectors by induction on $s = 3, ..., l$.

25. Lemma. *Suppose that $f \in C^{n-1}(U,Y)$ or $f \in C^{[n-1]}(U,Y)$, where U is open in $\mathbf{K}^m$. Then*

(1) $f \in C^n(U,Y)$ or $f \in C^{[n]}(U,Y)$ if and only if $\bar{\Phi}^n f(x; w, ..., w; t_1, ..., t_n)$ or
$\Upsilon^n f(x^{[n]})|_{\{U^{[n]}: v^{[k],i} = w_s \quad \forall 2^{s-1} < i \leq 2^s, 0 \leq s \leq k < n\}}$ is continuous for each marked $w \in \mathbf{K}^m$ or $w_0, ..., w_{n-1} \in \mathbf{K}^m$ respectively;

(2) $\bar{\Phi}^n f$ or $\Upsilon^n f$ is not locally bounded if and only if there exists marked $w \in \mathbf{K}^m$ or $w_0, ..., w_{n-1} \in \mathbf{K}^m$ such that $\bar{\Phi}^n f(x; w, ..., w; t_1, ..., t_n)$ or
$\Upsilon^n f(x^{[n]})|_{\{U^{[n]}: v^{[k],i} = w_s \quad \forall 2^{s-1} < i \leq 2^s, 0 \leq s \leq k < n\}}$ is not locally bounded.

Proof. In view of Lemma 11 and Formula 9(2) applied by induction we have

$\bar{\Phi}^n f(x; w, ..., w; t_1, ..., t_n) = \sum_{i_1, ..., i_n = 1}^m a_{i_1} ... a_{i_n} \bar{\Phi}^n f(x + t_1 \sum_{l_1 = i_1 + 1}^m a_{l_1} e_{i_1} + ... + t_n \sum_{l_n = i_n + 1}^m a_{l_n} e_{l_n}; e_{i_1}, ..., e_{i_n}; a_{i_1} t_1, ..., a_{i_n} t_n)$
for each $w = \sum_{i=1}^m a_i e_i$ if at least one $t_i \neq 0$, where $a_i \in \mathbf{K}$, for convenience of notation $\sum_{i=m+1}^m a_i e_i = 0$. Then consider all $t_1, ..., t_n \in \mathbf{K}$ such that $0 \neq t_i \to 0$. Due to Lemma 24 and since a_i are arbitrary and can be taken nonzero, then each $\bar{\Phi}^n f(x; e_{i_1}, ..., e_{i_n}; a_{i_1} t_1, ..., a_{i_n} t_n)$ is continuous or locally bounded if and only if $\bar{\Phi}^n f(x; w, ..., w; t_1, ..., t_n)$ is continuous or locally bounded for each marked $w \in \mathbf{K}^m$. In view of Lemma 21 this provides assertions $(1, 2)$ for $\bar{\Phi}^n f$.

We have $\hat{P}^n(x^{[n]})|_{(U^{[n]}: v^{[k],i} = w_s \forall 2^{s-1} < i \leq 2^s, 0 \leq s \leq k < n)} = x + \sum_{k=0}^{n-1} \phi_{k+1}(t) w_k$, where $\phi_l(t) = \sum_{1 \leq i_1 < ... < i_l \leq n} t_{i_1} ... t_{i_l}$ are linearly independent symmetric polynomials, $l = 1, ..., n$, $t = (t_1, ..., t_n)$, in particular, $\phi_1(t) = t_1 + ... + t_n$. Put $\alpha_{j,l} := a_{j,s}$ for each $j = 1, ..., m$, $2^{s-1} < l \leq 2^s$, $s = 0, ..., n-1$, where $w_s = \sum_{i=1}^m a_{i,s} e_i$ with $a_{i,s} \in \mathbf{K}$ for each $s = 0, ..., n-1$.

Applying Formula 11(2) by induction we get

$\Upsilon^n f(x^{[n]})|_{\{U^{[n]}: v^{[k],i} = w_s \quad \forall 2^{s-1} < i \leq 2^s, 0 \leq s \leq k < n\}} = \sum_{i_0, ..., i_{n-1} = 1}^m \sum_{1 \leq q_k \leq 2^k, k = 0, ..., n-1}$
$(\prod_{k=0}^{n-1} \alpha_{i_k, q_k}) \Upsilon^n f(x_J^{[n]})|_{\{U^{[n]}: v^{[s],l} = \delta_{l, q_s} e_{i_{s+1}}, \tau_{s+1} = \alpha_{i_s, q_s} t_{s+1} \quad \forall s = 0, ..., n-1, 1 \leq l \leq 2^s\}}$
for each marked w_s if at least one $t_i \neq 0$, where $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for each $i \neq j$, $J = \{(i_k, q_k) : k = 0, ..., n-1\}$; $\hat{\pi}^n(x_J^{[n]}) = \hat{P}^n(y)$, where τ_{k+1} corresponds to $x_J^{[n]}$ instead of t_{k+1} for $x^{[n]}$, $y \in (\mathbf{K}^m)^{[n]}$ corresponds to the set $(x; v^{[k],l} = \sum_{j_k \geq i_k + \delta_{l, q_k}} \alpha_{j_k, l} e_{j_k}, k = 0, ..., n-1, 0 \leq s \leq k < n, 2^{s-1} < l \leq 2^s; t_1, ..., t_n)$ in the notation introduced above. Then consider all $t_1, ..., t_n \in \mathbf{K}$ such that $0 \neq t_i \to 0$. Since $a_{i,s} \in \mathbf{K}$ are arbitrary constants which can be taken nonzero, then from Lemmas 21 and 24 the state-

ment of this lemma for $\Upsilon^n f$ as well follows, since $\Upsilon^1(f^{[n-1]}(x^{[n-1]}))(x^{[n]}) = [f^{[n-1]}(x^{[n-1]} + v^{[n-1]}t_n) - f^{[n-1]}(x^{[n-1]})]/t_n$ and $x^{[n]} = (x^{[n-1]}, v^{[n-1]}, t_n)$ and due to repeated application of Formula 24(1), and $g(h(z)e_i, y) \in C^0$ by $(z, y) \in U_1 \times U_2$ is equivalent to $g(ue_i, y) \in C^0$ by $(u, y) \in h(U_1) \times U_2$ for continuous function $h(z)$ by $z = (z_1, ..., z_a) \in U_1$, where U_1 and U_2 are domains in $\mathbf{K}^a$ and $\mathbf{K}^c$, $g(ue_i, y) \in Y$, $h(U_1) \subset \mathbf{K}$.

26. Lemma. *If $f \in C^{n-1}(U, Y)$ or $f \in C^{[n-1]}(U, Y)$, where U is open in $\mathbf{K}^m$. Then $\bar{\Phi}^n f(x^{(n)})|_{\{U^{(n)}:\exists i \quad |t_i|\ge\delta\}}$ or $\Upsilon^n f(x^{[n]})|_{\{U^{[n]}):\exists i \quad |t_i|\ge\delta, v_3^{[k]}=0 \forall k\}}$ is continuous respectively, where $\delta > 0$.*

Proof. Since $\bar{\Phi}^n f(x^{(n)}) = \bar{\Phi}^1(\bar{\Phi}^{n-1} f(x^{(n-1)}))(x^{(n)})$ and $\Upsilon^n f(x^{[n]}) = \Upsilon^1(\Upsilon^{n-1} f(x^{[n-1]}))(x^{[n]})$ whenever it exists and $\Upsilon^1 f(x^{[1]}) = \bar{\Phi}^1 f(x^{(1)}) = [f(x+vt) - f(x)]/t$ then in view of Lemmas 21 and 24 we get the statement of this lemma, since $\bar{\Phi}^{n-1} f(x^{(n-1)})$ or $\Upsilon^{n-1} f(x^{[n-1]})$ is continuous respectively and there is considered a domain with $|t_i| \ge \delta$ and $t_i + v_3^{[i]} t_{i+1} = t_i$, where $v_3^{[i]} = 0$.

27. Lemma. *Let $f : \mathbf{K}^b \to \mathbf{K}$ be a function such that $f \circ u \in C^{[n]}(\mathbf{K}, \mathbf{K})$ or $f \circ u \in C^n(\mathbf{K}, \mathbf{K})$ for $n \ge 0$ and $f \in C^{[n-1]}(\mathbf{K}^b, \mathbf{K})$ or $f \in C^{n-1}(\mathbf{K}^b, \mathbf{K})$ for $n \ge 1$ for each $u \in C^{[\infty]}(\mathbf{K}, \mathbf{K}^b)$ or $u \in C^\infty(\mathbf{K}, \mathbf{K}^b)$, where $\mathbf{K}$ is a field with a non-archimedean valuation and $2 \le b \in \mathbf{N}$, then $\Upsilon^n f(x^{[n]})$ or $\bar{\Phi}^n f(x^{(n)})$ respectively is a locally bounded function on $(\mathbf{K}^b)^{[n]}$ or $(\mathbf{K}^b)^{(n)}$ and f is continuous.*

Proof. At first prove, that f is continuous, when $n = 0$, since for $n \ge 1$ we have $C^0 \subset C^{n-1}$. Suppose the contrary, that there exists a sequence ${}_jz$ such that $\lim_{j\to\infty} {}_jz = z_0$ and a limit of the sequence $\{f({}_jz) : j\}$ either does not exist or is not equal to $f(z_0)$. Take c_j and r_j and $u(x)$ as above, then $\lim_{j\to\infty}(f \circ u)({}_jx) = \lim_{j\to\infty} f({}_jz) \ne f(z_0) = (f \circ u)(y_0)$, hence $f \circ u$ is not continuous at y_0 contradicting the assumption of this lemma.

Now suppose the contrary, that there exists $z_0^{[n]} \in (\mathbf{K}^b)^{[n]}$ or $z_0^{(n)} \in (\mathbf{K}^b)^{(n)}$ such that $\Upsilon^n f$ or $\bar{\Phi}^n f$ is unbounded in a neighborhood of $z_0^{[n]}$ or $z_0^{(n)}$ correspondingly. As a neighborhood take a ball $B((\mathbf{K}^b)^{[n]}, z_0^{[n]}, \epsilon)$ in $(\mathbf{K}^b)^{[n]}$ containing $z_0^{[n]}$ and of radius $\epsilon > 0$ or $B((\mathbf{K}^b)^{(n)}, z_0^{(n)}, \epsilon)$. Without loss of generality we may suppose, that $z_0 := z_0^{[0]} = 0 \in \mathbf{K}^b$ making the shift $\phi(x) := f(x - z_0)$ when $z_0 \ne 0$, where z_0 denotes the projection of $z_0^{[n]}$ in $\mathbf{K}^b$. Then there exists a sequence ${}_kz^{[n]}$ or ${}_kz^{(n)}$ tending to $z_0^{[n]}$ or $z_0^{(n)}$ when k tends to the infinity such that $\lim_{k\to\infty} |\Upsilon^n f({}_kz^{[n]})| = \infty$ or $\lim_{k\to\infty} |\bar{\Phi}^n f({}_kz^{(n)})| = \infty$ respectively, where $|x| = |x|_{\mathbf{K}}$ is the valuation in

$\mathbf{K}$. So we choose the sequence $\{ {_k}z_0^{[n]} : k = 1, 2, ...\}$ such that $| {_k}v^{[n-1]}| \le 1$ and $| {_k}t_j| \le 1$ for each $k \in \mathbf{N}$ and $j = 1, ..., n$. In view of Lemma 25 without loss of generality there exists a marked $w \in \mathbf{K}^m$ or $w_0, ..., w_{n-1} \in \mathbf{K}^m$ such that $\bar{\Phi}^n f(x; w, ..., w; t_1, ..., t_n)$ or $\Upsilon^n f(x^{[n]})|_{\{U^{[n]}: v^{[k],i} = w_s \quad \forall 2^{s-1} < i \le 2^s, 0 \le s \le k < n\}}$ is not locally bounded in a neighborhood of either $z_0^{(n)}$ or $z_0^{[n]}$ with the sequence $\{ {_j}z^{(n)} : j \in \mathbf{N}\}$ or $\{ {_j}z^{[n]} : j \in \mathbf{N}\}$ such that either $\{ {_j}z^{[n]} : j \in \mathbf{N}; {_j}v_i = w, i = 1, ..., n\}$ or $\{ {_j}z^{[n]} : j \in \mathbf{N}; {_j}v^{[k],i} = w_s \quad \forall 2^{s-1} < i \le 2^s, 0 \le s \le k < n\}$ respectively. At the same time due to Lemma 26 we can consider, that $\lim_{j\to\infty} \max_{i=1}^n | {_j}t_i| = 0$. From Formula 9(1) or 10(1) applied to $u = id$ and the conditions of this lemma it follows, that all terms with orders $k < n$ of $B_*^k f$ or A_*^k are continuous, hence there exists an ordered set $\{j_n, ..., j_1\}$ such that the sequence either

(1) $\{(B_{j_n,v^{(n-1)},t_n} ... B_{j_1,v^{(0)},t_1} f \circ u)(\bar{\Phi}^1 \circ p_{j_n} \hat{S}_{j_{n-1}+1,v^{(n-2)}t_{n-1}} ... \hat{S}_{j_1+1,v^{(0)}t_1} u^{n-1})(P_n \bar{\Phi}^1 \circ p_{j_{n-1}} \hat{S}_{j_{n-2}+1,v^{(n-3)},t_{n-2}} ... \hat{S}_{j_1+1,v^{(0)}t_1} u^{n-2}) ... (P_n ... P_2 \bar{\Phi}^1 \circ p_{j_1} u)({_j}z_0^{(n)}) : j \in \mathbf{N}\}$ or

(2) $\{(A_{j_n,v^{[n-1]},t_n} ... A_{j_1,v^{[0]},t_1} f \circ u)(\Upsilon^1 \circ p_{j_n} \hat{S}_{j_{n-1}+1,v^{[n-2]}t_{n-1}} ... \hat{S}_{j_1+1,v^{[0]}t_1} u^{n-1})(P_n \Upsilon^1 \circ p_{j_{n-1}} \hat{S}_{j_{n-2}+1,v^{[n-3]}t_{n-2}} ... \hat{S}_{j_1+1,v^{[0]}t_1} u^{n-2}) ... (P_n ... P_2 \Upsilon^1 \circ p_{j_1} u)({_j}z_0^{[n]}) : j \in \mathbf{N}\}$ is unbounded for $f = id$.

Now consider the same Formulas 10(1) or 9(1) for arbitrary u satisfying conditions of this lemma. Again all terms with orders $k < n$ of $B_*^k f \circ u$ or $A_*^k f \circ u$ are continuous and hence bounded in a neighborhood of $z_0^{(n)}$ or $z_0^{[n]}$ respectively. We construct a curve u in several steps leading to the contradiction with the supposition of this lemma.

Mention that $\Upsilon^1 id(y, v^{[0]}, t_1) = v^{[0]}$, where $y, v^{[0]} \in \mathbf{K}^b$ and $t_1 \in \mathbf{K}$. Then $\Upsilon^2 id(y^{[2]}) = (v^{[0]} + v_2^{[1]} t_2 - v^{[0]})/t_2 = v_2^{[1]}$ and $\Upsilon^3 id_j(y^{[3]}) = ({_j}v_2^{[1]} + {_{j+b}}v_2^{[2]} t_3 - v_2^{[1]})/t_3 = {_{j+b}}v_2^{[2]}$, where $j = 1, ..., b$, $v^{[k]} = ({_1}v_1^{[k]}, ..., {_c}v_1^{[k]}, {_1}v_2^{[k]}, ..., {_c}v_2^{[k]}, v_3^{[k]})$, $c = c(k) = 2^{k-1} - k + b(2^k - 1)$, ${_j}v_l^{[k]} \in \mathbf{K}$ for each j, k, l, $id(y) = (id_1(y), ..., id_b(y)) = (y_1, ..., y_b)$. Therefore, we get by induction

$\Upsilon^m id_j(y^{[m]}) = {_{j(m)}}v_2^{[m-1]}$,

for each $m \ge 2$, where $j(1) = j$, $j(2) = j$, $j(3) = j + b$, $j(m) = j + 2^{m-2} - (m-1) + b(2^{m-1} - 1)$ for each $m \ge 4$, since $j(m) = j + b + (2b+1) + (2(2b+1)+1) + (2(2(2b+1)+1)+1) + ... + (2(2(...(2b+1)+1)+1)$ with 2 in power $m-3$ in the latter term.

At first consider equations

(3) $\bar{\Phi}^k u(x^{(n)}) = \alpha_k \bar{\Phi}^k id(z^{(n)})$ or

(4) $\Upsilon^k u(x^{[n]}) = \alpha_k \Upsilon^k id(z^{[n]})$ for $k=0,1,...,n$ in neighborhoods of $x_0^{(n)}$ and $z_0^{(n)}$ or $x_0^{[n]}$ and $z_0^{[n]}$ with prescribed marked vectors η or $\eta_0,...,\eta_{n-1}$ and w or $w_0,...,w_{n-1}$ respectively, where η or $\eta_0,...,\eta_{n-1}$ are determined from the equations, $0\ne\alpha_k\in\mathbf{K}$ are constants specified below for a sequence such that $\lim_{j\to\infty} g_j = 0$, where $0<q_j := \min_{k=1}^n |\alpha_{j,k}| \le g_j := \max_{k=1}^n |\alpha_{j,k}| < 1$. If $t_s = 0$, then equations for $\Upsilon^k u$ simplify due to term D_{t_s} instead of $\Upsilon^1_{t_s}$ for which w_s does not play a role and we can consider $\tau_s = 0$, where τ_s play the same role for $x^{(n)}$ and $x^{[n]}$ as t_s for $z^{(n)}$ and $z^{[n]}$, $s=1,...,n$. If $t_s\ne 0$, then we can take $\tau_s\ne 0$. In view of Lemma 22 we can consider the data $(b-1,b)$ instead of $(1,b)$. Since w or $w_0,...,w_{n-1}$ are fixed vectors independent of j, then we can resolve these equations for marked nonzero vectors $\eta\in\mathbf{K}^{b-1}$ or $\eta_0,...,\eta_{n-1}\in\mathbf{K}^{b-1}$ corresponding to ${}_jx^{(n)}$ or ${}_jx^{[n]}$ such that variables will be ${}_jx\in\mathbf{K}^{b-1}$ and $\tau_1,...,\tau_n$ for u instead of ${}_jz\in\mathbf{K}^b$ and $t_1,...,t_n$ for f, such that $\lim_{j\to\infty}\max_{i=1}^n|\tau_i| = 0$. In view of Formulas 12(2) and 14(1) it is sufficient to consider a quadratic function

(5) $u(h) = z + c\sum_{k_1,k_2=0}^2 \sum_{i_1,i_2=1}^{b-1} {}_{i_1,i_2}a_{k_1,k_2} h_{i_1}^{k_1} h_{i_2}^{k_2}$,

where ${}_{i_1,i_2}a_{k_1,k_2}\in\mathbf{K}^b$, $c\in\mathbf{K}$, $|{}_{i_1,i_2}a_{k_1,k_2}|\le 1$ for each i_1,i_2,k_1,k_2, $h=(h_1,...,h_{b-1})\in\mathbf{K}^{b-1}$. Thus we get

(6) $|(B_{j_n,\eta^{\otimes n},\tau_n}...B_{j_1,\eta,\tau_1}f\circ u)(\bar\Phi^1\circ p_{j_n}\hat S_{j_{n-1}+1,\eta^{\otimes(n-1)}\tau_{n-1}}$
$...\hat S_{j_1+1,\eta\tau_1}u^{n-1})(P_n\bar\Phi^1\circ p_{j_{n-1}}\hat S_{j_{n-2}+1,\eta^{\otimes(n-2)},\tau_{n-2}}...\hat S_{j_1+1,\eta\tau_1}u^{n-2})$
$...(P_n...P_2\bar\Phi^1\circ p_{j_1}u)({}_jx_0^{(n)})| \ge |q_j|^n|\pi|^{l_0+s_0}|(B_{j_n,w^{\otimes n},t_n}...B_{j_1,w,t_1}f\circ id)(\bar\Phi^1\circ$
$p_{j_n}\hat S_{j_{n-1}+1,w^{\otimes(n-1)}t_{n-1}}$
$...\hat S_{j_1+1,wt_1}id^{n-1})(P_n\bar\Phi^1\circ p_{j_{n-1}}\hat S_{j_{n-2}+1,w^{\otimes(n-2)},t_{n-2}}...\hat S_{j_1+1,wt_1}id^{n-2})$
$...(P_n...P_2\bar\Phi^1\circ p_{j_1}id)({}_jz_0^{(n)})|$ or

(7) $|(A_{j_n,\eta^{[n-1]},\tau_n}...A_{j_1,\eta^{[0]},\tau_1}f\circ u)(\Upsilon^1\circ p_{j_n}\hat S_{j_{n-1}+1,\eta^{[n-2]}\tau_{n-1}}$
$...\hat S_{j_1+1,\eta^{[0]}\tau_1}u^{n-1})(P_n\Upsilon^1\circ p_{j_{n-1}}\hat S_{j_{n-2}+1,\eta^{[n-3]}\tau_{n-2}}...\hat S_{j_1+1,\eta^{[0]}\tau_1}u^{n-2})$
$...(P_n...P_2\Upsilon^1\circ p_{j_1}u)({}_jx_0^{[n]})| \ge |q_j|^n|\pi|^{l_0+s_0}|(A_{j_n,w^{[n-1]},t_n}...A_{j_1,w^{[0]},t_1}f\circ id)(\Upsilon^1\circ$
$p_{j_n}\hat S_{j_{n-1}+1,w^{[n-2]}t_{n-1}}$
$...\hat S_{j_1+1,w^{[0]}t_1}id^{n-1})(P_n\Upsilon^1\circ p_{j_{n-1}}\hat S_{j_{n-2}+1,w^{[n-3]}t_{n-2}}...\hat S_{j_1+1,w^{[0]}t_1}id^{n-2})$
$...(P_n...P_2\Upsilon^1\circ p_{j_1}id)({}_jz_0^{[n]})|$

for each $j\in\mathbf{N}$, where $l_0\in\mathbf{N}$ is a marked number, $s_0=s_0(j)\in\mathbf{N}$, each $w^{[k]}$ corresponds to marked $w_0,...,w_{n-1}$, while $\eta,\eta_0,...,\eta_{n-1}\in\mathbf{K}^{b-1}$ are marked vectors for u, where $w^{\otimes k} := (w,...,w)\in X^{\otimes k}$ for $w\in X$ and $k\in\mathbf{N}$.

Take a function $\psi\in C^\infty(\mathbf{K},\mathbf{K})$ such that $\psi(x)=1$ for $|x|\le|\pi|$ and

$\psi(x) = 0$, when $|x| > |\pi|$, for example, locally constant function, where $\pi \in \mathbf{K}$, $0 < |\pi| < 1$. In particular, the characteristic function of $B(\mathbf{K}, 0, |\pi|)$ is locally analytic, since $\mathbf{K}$ is totally disconnected with the base of its topology consisting of clopen (closed and open simultaneously) balls, where $B(X, x, R) := \{y \in X, \rho(x, y) \le R\}$ for a topological space X metrizable by a metric ρ. It is proved further that such ψ after definite scalings suits construction below. Define now the functions

$u_j(h) := (\xi_j\psi)((h - {}_jx)/T_j)$, where

$\xi_j(h) := [\,{}_{r_j}z_0 + c_j \sum_{k_1,k_2=0}^{2} \sum_{i_1,i_2=1}^{b-1} {}_{i_1,i_2}a_{k_1,k_2} h_{i_1}^{k_1} h_{i_2}^{k_2}]$ such that $\xi_j(0) = {}_{r_j}z_0$ and put

$u(x) := \sum_{j=1}^{\infty} u_j(x)$,

where $x = (x_1, ..., x_{b-1}) \in \mathbf{K}^{b-1}$, each ${}_{i_1,i_2}a_{k_1,k_2} \in \mathbf{K}^b$ is marked, $c_j \in \mathbf{K}\setminus\{0\}$. Choose $r_j \in \mathbf{N}$, ${}_jx_i, T_j \in \mathbf{K}$ later on. All u_j have disjoint supports, hence the series is convergent, u is of class $C^{[\infty]} := \bigcap_{n=1}^{\infty} C^{[n]}$ in $\mathbf{K} \setminus \{z_0\}$.

Consider the sets $\lambda_i := \{j \in \mathbf{N} : {}_jt_i = 0\}$, then either $card(\mathbf{N}\setminus\lambda_i) = \aleph_0$ or $card(\lambda_i) = \aleph_0$ or both cardinalities are $\aleph_0$. Consider intersections $A_1 \cap ... \cap A_n$, where $A_i = \lambda_i$ or $A_i = \mathbf{N} \setminus \lambda_i$. The union of all such finite intersections is $\mathbf{N}$. Therefore, one of these intersections is of the cardinality $\aleph_0$. Thus, there exists a subsequence $\{j(l) : l \in \mathbf{N}\}$ such that ${}_st_{j(l)} = 0$ for each l and every $s \in \{i_1, ..., i_r\}$ and ${}_st_{j(l)} \ne 0$ for each $s \in \{1, ..., n\} \setminus \{i_1, ..., i_r\}$, where $0 \le r \le n$. After the enumeration we can consider a sequence with such property. For such a sequence we can choose a subsequence which after enumeration has the property:

(8) $|\,{}_{j+1}t_i| \le |\pi|^{s(j)}|\,{}_jt_i|$ and

(9) $|\pi|^{r(j)} b_{j+1} \ge b_j$ for each $j \in \mathbf{N}$ and $i \in \{1, ..., n\}$,

where $s(j), r(j) \in \mathbf{N}$ are sequences specified below;

$b_j := |(B_{j_n,w^{\otimes n},t_n} ... B_{j_1,w,t_1} f \circ id)(\bar{\Phi}^1 \circ p_{j_n} \hat{S}_{j_{n-1}+1,w^{\otimes(n-1)}t_{n-1}} ... \hat{S}_{j_1+1,w\tau_1} id^{n-1})(P_n \bar{\Phi}^1 \circ p_{j_{n-1}} \hat{S}_{j_{n-2}+1,w^{\otimes(n-2)},t_{n-2}} ... \hat{S}_{j_1+1,wt_1} id^{n-2}) ... (P_n ... P_2 \bar{\Phi}^1 \circ p_{j_1} id)({}_jz_0^{(n)})|$

or with analogous Properties $(8, 9)$ for $A_*^n f$ instead of $B_*^n f$.

Now choose r_j and c_j such that $\lim_{j\to\infty} c_j T_j^{-q} = 0$ for each $q \in \mathbf{N}$, for example, $c_j = T_j^j$, where $\lim_{j\to\infty} T_j = 0$, $|T_j| > |T_{j+1}|$ for each j, $T_j \ne 0$ for each j. Then choose $r_j \in \mathbf{N}$ such that $\max_{l=1}^{n}(|\,{}_{r_j}t_l|) \le |c_j|$ and $|\,{}_{r_j}z_0|) \le |c_j|$ for each j and $\lim_{j\to\infty} |c_j^n \Upsilon^n f({}_{r_j}z_0^{[n]})| = \infty$. Take ${}_jx \in \mathbf{K}^{b-1}$ such that ${}_jx_i = (\pi^{-1} \sum_{k=1}^{j-1} T_k) + T_j$, where $\pi \in \mathbf{K}$, $0 < |\pi| < 1$, $i = 1, ..., b-1$. Since $|T_j| > |T_{j+1}| > 0$ for each $j \in \mathbf{N}$, then $|\,{}_jx - {}_{j+1}x| = |T_{j+1} + T_j(\pi^{-1} - 1)| =$

$|\pi^{-1}T_j| > |T_j|$ and $|\ _kx - \ _{k+1}x| \ge \min(|\ _kx - \ _{k+1}x|, |\ _{k+1}x - \ _{k+2}x|, ..., |\ _jx - \ _{j+1}x|) \ge \min(|T_k|, ..., |T_j|)$ for each $k \le j$, consequently, $B(\mathbf{K}^{b-1}, \ _kx, |T_k|) \cap B(\mathbf{K}^{b-1}, \ _{j+1}x, |T_{j+1}|) = \emptyset$ for each $k \le j$, hence $supp(u_j) \cap supp(u_k) = \emptyset$ for each $k < j$. Take $s_0(j+1) \ge s_0(j) + j + 1$ and $|\pi|^{s_0(j+1)} < |q_j| \le g_j \le |\pi|^{s_0(j)}$ and $r(j) \ge s_0(j)2n$ and $s(j) \ge s_0(j)$ for each j, where l_0 is such that $0 < |\pi|^{l_0} < 1/2$ (see also $(6-9)$).

Denote $y_0 := \lim_{j\to\infty} \ _jx$. Then u is of class $C^{[\infty]}_b$ or C^∞_b in a neighborhood of z_0. To prove this we show, that $\Upsilon^q u(z^{[q]})$ or $\bar{\Phi}^q u(z^{(q)})$ tends to zero as z tends to $z_0 = 0$, where $|z^{[q]}| < \epsilon$ or $|z^{(q)}| < \epsilon$, since then for $|t_1| \ge \epsilon$, or $..., |t_n| \ge \epsilon$, $|v^{[q-1]}| \le 1$ the continuity will be evident. For this we use Lemma 15. Mention, that $\|\Upsilon^q\psi(x)\|_{C^0(B(\mathbf{K}^{[q]},0,R),\mathbf{K})} < \infty$ for each q and each $R > 0$. Indeed, $\max_{x\in\mathbf{K}} |\psi(x)| = 1$ so that $\Upsilon^0\psi$ is bounded for $q = 0$. For $q = 1$ we have $\Upsilon^1\psi(x, v, t_1) = 0$ for $\max(|x|, |x+vt_1|) \le |\pi|$ or $\min(|x|, |x+vt_1|) > |\pi|$, $\Upsilon^1\psi(x, v, t_1) = 1/t_1$ for either $|x| \le |\pi|$ and $|x + vt_1| > |\pi|$ or $|x| > |\pi|$ and $|x + vt_1| \le |\pi|$. Since we consider the domain $|x^{[1]}| \le R$, then $|v| \le R$, consequently, $\|\Upsilon^1\psi(x)\|_{C^0(B(\mathbf{K}^{[1]},0,R),\mathbf{K})} \le R|\pi|^{-1}$, since $|t_1|^{-1} \le |\pi|^{-1}R$ in the considered domain, when $\Upsilon^1\psi(x, v, t_1) \ne 0$. The function $\Upsilon^1\psi(x, v, t_1)$ is the product of the locally constant function by variables (x, v) and the function $1/t_1$ with $|\pi/v| \le |t_1| \le R$, when this function is nonzero and $v \ne 0$, hence $|\pi|/R \le |v| \le R$, that is $|\pi|/R \le |t_1| \le R$, where $\Upsilon^1\psi(x, 0, t_1) = 0$ for each x and t_1. Evidently, by induction that $\Upsilon^q\psi(x^{[q]})$ is in $C^0(B(\mathbf{K}^{[q]}, 0, R), \mathbf{K})$ with the finite norm $\|\Upsilon^q\psi(x^{[q]})\|_{C^0(B(\mathbf{K}^{[q]},0,R),\mathbf{K})} \le C^{q+1}V_q^{-q} \le (q+1)(R/|\pi|)^q$ with $V_q = 1$ and $C := \lim_{q\to\infty}[(q+1)(R/|\pi|)^q]^{1/(q+1)}$ for non-scaled ψ for each $q \in \mathbf{N}$ and each $R \ge 1$. In general for scaled ψ put $V_q := \min_{j=1}^q |T_j| > 0$. At the same time for each x with $|x - \ _jx| \le |T_j|$ and $|v^{[k]}| \le R$ and $|t_{k+1}| \le R$ for each $k = 0, ..., n-1$ in accordance with Lemmas 4, 12, 15 and Corollary 13

(10) $|\Upsilon^q u(x^{[q]})| \le (\max(1, R^2))|c_j||T_j|^{-q}C_1^{q+1}V_q^{-q}$

which tends to zero as j tends to the infinity, since $C_1^{q+1} \le (q+1)(R/|\pi|)^q$, $0 < |T_{j+1}| < |T_j|$ for each j and $\lim_{j\to\infty} c_jT_j^{-\beta} = 0$ for every $\beta \in \mathbf{N}$, where $R \ge 1$.

If each term in Formula 9(1) OR 10(1) would be locally bounded, then $\bar{\Phi}^n(f \circ u)(\ _{r_j}x^{(n)})$ or $\Upsilon^n(f \circ u)(\ _{r_j}x^{[n]})$ would be locally bounded. Since each $\bar{\Phi}^k f$ or $\Upsilon^k f$ is locally bounded for $k < n$ by our supposition above, then from Formula 9(1) or 10(1) and the condition $\lim_{j\to\infty} |c_j^n\Upsilon^n f(\ _{r_j}z_0^{[n]})| = \infty$ or $\lim_{j\to\infty} |c_j^n\bar{\Phi}^n f(\ _{r_j}z_0^{(n)})| = \infty$ it follows, that there exists a term or a finite

sum of terms of the type

$(A_{j_n,v^{[n-1]},t_n}...A_{j_1,v^{[0]},t_1}f\circ u)(\Upsilon^1\circ p_{j_n}S_{j_{n-1}+1,v^{[n-2]}t_{n-1}}$
$...S_{j_1+1,v^{[0]}t_1}u^{n-1})(P_n\Upsilon^1\circ p_{j_{n-1}}S_{j_{n-2}+1,v^{[n-3]}t_{n-2}}...S_{j_1+1,v^{[0]}t_1}u^{n-2})...(P_n...P_2\Upsilon^1\circ$
$p_{j_1}u)\ {}_{r_l}x^{[n]})$

which absolute value tends to the infinity for a particular set ω of indices $(j_1,...,j_n)$ and a subsequence $\{\ {}_{r_l}x^{[n]}: j\in \mathbf{N}\}$ or analogously for $B^n_*f\circ u$ instead of $A^n_*f\circ u$. But this contradicts supposition of this lemma in view of Lemmas 9, 21 and Corollary 10. Therefore, $\Upsilon^n f$ or $\bar{\Phi}^n f$ respectively is locally bounded.

28. Remark. Though $u\in C^\infty(\mathbf{K},\mathbf{K}^b)$, but u is not locally analytic in general, since the sequence $\{x_j: j\}$ converges to $y_0\in\mathbf{K}$ and u has not a series expansion in a neighborhood of y_0 with positive radius of convergence.

29. Definitions. Let $\phi:(0,\infty)\to(0,\infty)$ be a function such that $\lim_{q\to 0}\phi(q)=0$. By either $\mathbf{K}(\phi)$ or $\mathbf{K}(u,\phi)$ we denote the $\mathbf{K}$-linear space of all functions $f:\mathbf{K}^m\to\mathbf{K}$ such that for each bounded subset U in $\mathbf{K}^m$ there exists a constant $C>0$ such that either

(1) $|f(x+y)-f(x)|\le C\phi(|y|)$,

when $x\in U$ and $x+y\in U$ or

(2) $|f(x+ut)-f(x)|\le C\phi(|t|)$,

when $x\in U$ and $x+ut\in U$ respectively, where $u\in\mathbf{K}^m$ is a nonzero vector. In the particular case of $\phi(q)=q^w$, where $0<w\le 1$, we also denote $\mathbf{K}(q^w)=: Lip(w)$ and $\mathbf{K}(u,q^w)=: Lip(u,w)$.

Then we denote by $C^{[n],w}(\mathbf{K}^m,\mathbf{K})$ or $C^{[n]}_\phi(\mathbf{K}^m,\mathbf{K})$ or $C^{n,w}(\mathbf{K}^m,\mathbf{K})$ or $C^n_\phi(\mathbf{K}^m,\mathbf{K})$ the $\mathbf{K}$-linear space of all functions $f\in C^{[n]}(\mathbf{K}^m,\mathbf{K})$ in the first and the second cases or in $C^n(\mathbf{K}^m,\mathbf{K})$ in the third and the fourth cases such that $f^{[n]}(x^{[n]})\in Lip(w)$ or $f^{[n]}(x^{[n]})\in\mathbf{K}(\phi)$ or $\bar{\Phi}^n f(x^{(n)})\in Lip(w)$ or $\bar{\Phi}^n f(x^{(n)})\in\mathbf{K}(\phi)$ respectively.

30. Lemma. *Let suppositions of Lemma 27 be satisfied and moreover $\Upsilon^n(f\circ u)\in\mathbf{K}(\phi)$ or $\bar{\Phi}^n(f\circ u)\in\mathbf{K}(\phi)$ for each $u\in C^{[\infty]}(\mathbf{K},\mathbf{K}^b)$ or $C^{[\infty]}_b(\mathbf{K},\mathbf{K}^b)$ or $u\in C^\infty(\mathbf{K},\mathbf{K}^b)$ or $C^\infty_b(\mathbf{K},\mathbf{K}^b)$, then $\Upsilon^n f(x^{[n]})\in\mathbf{K}(v,\phi)$ or $\bar{\Phi}^n f(x^{(n)})\in\mathbf{K}(v,\phi)$, where v is a marked vector $v\in(\mathbf{K}^b)^{[n]}$ or $v\in(\mathbf{K}^b)^{(n)}$, where $x\in\mathbf{K}^b$, $x^{[n]}\in(\mathbf{K}^b)^{[n]}$ or $x^{(n)}\in(\mathbf{K}^b)^{(n)}$ correspondingly.*

Proof. Without loss of generality it can be assumed, that the function ϕ is subadditive and increasing taking

$\phi_1(q):=\inf\{\sum_{k=1}^n\phi(q_k):\quad \sum_{k=1}^n q_k\ge q, q_k\ge 0\}$, which is the largest increasing and subadditive minorant of ϕ. For the subadditive and increasing

ϕ there is satisfied the inequality:

(1) $\phi(q\epsilon) \le \phi((1+[q])\epsilon) \le (1+q)\phi(\epsilon)$

for each $\epsilon > 0$ and $q > 0$, where $[q]$ denotes the integral part of q such that $[q] \le q$.

If S is a family of vectors such that it spans $\mathbf{K}^b$ and f belongs to $\mathbf{K}(u,\phi)$ for each $u \in S$, then $f \in \mathbf{K}(\phi)$, since $b \in \mathbf{N}$ and $|f(x+y) - f(x)| = |f(x+y) - f(x+y_2e_2+...+y_be_b) + f(x+y_2e_2+...+y_be_b) - f(x+y_3e_3+...+y_be_b) + ... + f(x+y_be_b) - f(x)| \le \max(|f(x+y) - f(x+y_2e_2+...+y_be_b)|, |f(x+y_2e_2+...+y_be_b) - f(x+y_3e_3+...+y_be_b)|, ..., |f(x+y_be_b) - f(x)|) \le C\max(\phi(|y_1|), ..., \phi(|y_b|)) \le C\phi(|y|)$ due to increasing monotonicity of ϕ and the fact that $|y| = \max(|y_1|, ..., |y_b|)$, where $y = y_1e_1 + ... + y_be_b$, $y_1, ..., y_b \in \mathbf{K}$, $e_j = (0, ..., 0, 1, 0, ...) \in \mathbf{K}^b$ with 1 on the j-th place and up to a $\mathbf{K}$-linear topological automorphism of $\mathbf{K}^b$ onto itself we can choose such basis as belonging to S, $j = 1, ..., b$, and $C > 0$ is a constant.

Let us assume that for some point the statement of this lemma is not true. We can suppose, that this is at $x^{[n]} = (0, ..., 0) \in (\mathbf{K}^b)^{[n]}$ or $x^{(n)} = 0 \in (\mathbf{K}^b)^{(n)}$ respectively making a shift in a case of necessity. Then there exist sequences $b_k > 0$, $h_k \in \mathbf{K}$, $h_k \ne 0$, ${}_kz^{[n]} \in (\mathbf{K}^b)^{[n]}$ such that $\lim_{k\to\infty} b_k = \infty$, $\lim_{k\to\infty} h_k = 0$, $\lim_{k\to\infty} {}_kz^{[n]} = 0$ and

(2) $|\Upsilon^n f({}_kz^{[n]} + h_kv) - \Upsilon^n f({}_kz^{[n]})| > b_k\phi(|h_k|)$ or

(2′) $|\bar{\Phi}^n f({}_kz^{(n)} + h_kv) - \bar{\Phi}^n f({}_kz^{(n)})| > b_k\phi(|h_k|)$

with $\lim_{k\to\infty} {}_kz^{(n)} = 0$ respectively, where $0 \ne v \in (\mathbf{K}^b)^{[n]}$ or $0 \ne v \in (\mathbf{K}^b)^{(n)}$ correspondingly, $k = 1, 2, 3, ...$. Let the functions u and u_j be as in the proof of Lemma 27. Choose $r_j \in \mathbf{N}$ such that $|{}_{r_j}z_0| \le |c_j|$, $\lim_{j\to\infty} |c_j|^{n+1}b_{r_j} = \infty$, $|h_{r_j}| < |\pi c_jT_j|$. Thus $u \in C^\infty(\mathbf{K}, \mathbf{K}^b)$. Now prove that at least for large $j \in \mathbf{N}$ there is accomplished the inequality:

(3) $|\Upsilon^n(f\circ u)({}_jx^{[n]} + {}_j\nu^{[n]}) - \Upsilon^n(f\circ u)({}_jx^{[n]})| > |\pi c_j^{n+1}|b_{r_j}\phi(|{}_j\nu^{[n]}|)|\pi|^{l_0}$

or

(3′) $|\bar{\Phi}^n(f\circ u)({}_jx^{(n)} + {}_j\nu^{(n)}) - \bar{\Phi}^n(f\circ u)({}_jx^{(n)})| > |\pi c_j^{n+1}|b_{r_j}\phi(|{}_j\nu^{[n]}|)|\pi|^{l_0}$,

where $|{}_j\nu^{[n]}| = |h_{r_j}v/c_j|$ or $|{}_j\nu^{(n)}| = |h_{r_j}v/c_j|$ with $c_j \ne 0$ for each j. Take without loss of generality $|v| = 1$. Together with the condition $\lim_{j\to\infty} |c_j|^{n+1}b_{r_j} = \infty$ this will complete the proof. If $|h| < |\pi T_j|$, then

(4) $u_j(h) = {}_{r_j}z_0 + c_j\sum_{k_1,k_2=0}^{2}\sum_{i_1,i_2=1}^{b-1} {}_{i_1,i_2}a_{k_1,k_2}h_{i_1}^{k_1}h_{i_2}^{k_2}$ with $u_j(0) = {}_{r_j}z_0$.

In formula 9(1) or 10(1) all terms with an amount of operators $A_{j,v^{[k-1]},t_k}$ or $B_{j,v^{[k-1]},t_k}$ in it less than n are in $C^{[1]}_\phi(\mathbf{K}, \mathbf{K})$. As in Lemma 27 reduce the consideration to $\Upsilon^n f({}_kz^{[n]})$ or $\bar{\Phi}^n f({}_kz^{(n)})$ with prescribed fixed vectors

$w_0, w_1, ..., w_{n-1}$ with $v^{[0]} = v^{[k],1} = w_0$ and $v^{[k],i} = w_l$ for each $2^{l-1} < i \le 2^l$, where $l = 0, 1, ..., k$ and $k = 0, 1, ..., n-1$, vectors $v^{[k],i} \in \mathbf{K}^b$ are formed from $v^{[k]}$ after excluding all zeros arising from $v_3^{[k]} = 0$, or $v = (w_0, ..., w_0)$ with ${}_kz^{(n)} = ({}_kz_0; v; {}_kt)$ respectively, where ${}_kt = ({}_kt_1, ..., {}_kt_n) \in \mathbf{K}^n$. In view of Lemma 16 we can consider the case $(b-1, b)$ instead of $(1, b)$. Thus, from Formulas $(2,4)$ and 12(2) it follows, that there exist expansion coefficients ${}_{i_1,i_2}a_{k_1,k_2} \in \mathbf{K}^b$ with $|{}_{i_1,i_2}a_{k_1,k_2}| \le 1$ for each i_1, i_2, k_1, k_2 and there exists $j_0 \in \mathbf{N}$, for which

(5) $|\Upsilon^n f \circ u({}_jx^{[n]} + {}_j\nu^{[n]}) - \Upsilon^n f \circ u({}_jx^{[n]})| \ge |\pi|^{l_0+s_0}|q_j|^n$
$|\Upsilon^n f({}_{r_j}z_0^{[n]} + h_jv) - \Upsilon^n f({}_{r_j}z_0^{[n]})| \ge b_{r_j}|c_j^n|\phi(|c_j\, {}_j\nu^{[n]}|)|\pi|^{l_0}$ or

(5') $|\bar{\Phi}^n f \circ u({}_jx^{(n)} + {}_j\nu^{(n)}) - \bar{\Phi}^n f \circ u({}_jx^{(n)})| \ge |\pi|^{l_0+s_0}|q_j|^n$
$|\bar{\Phi}^n f({}_{r_j}z_0^{(n)} + h_jv) - \bar{\Phi}^n f({}_{r_j}z_0^{(n)})| \ge b_{r_j}|c_j^n|\phi(|c_j\, {}_j\nu^{(n)}|)|\pi|^{l_0}$

for each $j \ge j_0$, since $|\nu_j| < |\pi T_j|$, where $l_0 \in \mathbf{N}$ is a marked natural number, ${}_j\tau_i$, $i = 1, ..., n$ are parameters corresponding to $t_1, ..., t_n$, but for the curve u instead of f. There exists $j_0 \in \mathbf{N}$ such that $|c_j| \le \min(1, |\pi|^{-1} - 1)$ for each $j > j_0$, where $\pi \in \mathbf{K}$, $0 < |\pi| < 1$. In view of Formula (1) for each $j > j_0$ we have $\phi(|{}_j\nu^{[n]}|) \le (1 + |c_j|^{-1})\phi(|c_j\, {}_j\nu^{[n]}|) \le |\pi c_j|^{-1}\phi(|c_j\, {}_j\nu^{[n]}|)$. Therefore, the latter formula and Formula (5) imply Formula (3).

31. Lemma. *Let f be a function $f : \mathbf{K} \to \mathbf{K}$ such that $f(0) = 0$ and $|f(t)| \le 1$ for each $t \in \mathbf{K}$ such that $|t| \le |q|a$, where q and a are constants such that $q \in \mathbf{K}$, $|q| > 1$, $a > 0$, and assume that*

(1) $|f(qt) - qf(t)| \le \max(b, C_1|t|^r)$

for each $t \in \mathbf{K}$ with $|t| \le a$, where $0 < r \le 1$, $b > 0$ and $C_1 > 0$ are constants. Then there exists a constant $C_2 > 0$ such that

(2) $|f(t)| \le \max(b, C_2|t|^r)$

for each $t \in \mathbf{K}$ with $|t| \le |q|a$, where $C_2 = \max(a^{-r}, |q|^{-1}a^rC_1|q|^{-r})$.

Proof. If $t \in \mathbf{K}$ is such that $a \le |t| \le |q|a$, then Inequality (2) is satisfied with $C_2 = a^{-r}$, since $|f(t)| \le 1$ for such t. Now suppose that $0 < |u| < a$ and Inequality (2) is satisfied for $t = qu$, then

$|q||f(t)| \le \max(|f(qt)|, b, C_1|u|^r) \le \max(b, C_2|qu|^r, C_1|u|^r)$
$= \max(b, |u|^r \max(C_1, C_2|q|^r))$, hence

$|f(t)| \le |q|^{-1}\max(b, a^r\max(C_1, C_2|q|^r)) \le \max(b, C_2|t|^r)$

for $C_2 = \max(a^{-r}, |q|^{-1}a^rC_1|q|^{-r})$, since $C_2|q|^r \ge C_1$. On the other hand $B(\mathbf{K}, 0, |q|a) \setminus \{0\}$ is the disjoint union of subsets $B(\mathbf{K}, 0, |q|^ja) \setminus B(\mathbf{K}, 0, |q|^{j-1}a)$ for $j = 1, 0, -1, -2, ...$ Therefore, proceeding by induction by j we get the statement of this lemma, since $f(0) = 0$.

32. Lemma. *Let Ω be a finite set of vectors $v \in \mathbf{K}^m$ which are pairwise $\mathbf{K}$-linearly independent, $card(\Omega) \ge m$ and each subset of Ω consisting not less than m vectors has the $\mathbf{K}$-linear span coinciding with $\mathbf{K}^m$, where $m \ge 2$ is the integer. Suppose that for each $v \in \Omega$ there is given a function $g_v : \mathbf{K}^m \to \mathbf{K}$ such that:*

(1) $|g_v(x)| \le 1$ *for each $x \in \mathbf{K}^m$ with $|x| \le R$,*

(2) $|g_v(x+tv) - g_v(x)| \le |t|^r$ *for each $x, x+tv \in \mathbf{K}^m$ with $|x| \le R$ and $|x+tv| \le R$, where $t \in \mathbf{K}$,*

(3) $|\sum_{v \in \Omega}(g_v(x) - g_v(y))| \le b$ *for each $|x| \le R$ and $|y| \le R$, where R and b are positive constants, $0 < r \le 1$. Then there exists a constant C, which may depend on (r, Ω) such that*

(4) $|g_v(x) - g_v(y)| \le C \max(b, |x-y|^r)$
for each $x, y \in \mathbf{K}^m$ such that $|x| \le R$, $|y| \le R$ and each $v \in \Omega$.

Proof. Prove this lemma by induction on a number n of elements in Ω. For $n = m = 1$ Inequality (4) is the consequence of Inequality (2). For $n = m \ge 2$ vectors $v_1, ..., v_m$ by the supposition of lemma are $\mathbf{K}$-linearly independent. Then for each $x, y \in \mathbf{K}^m$ there exist $t_1, ..., t_m \in \mathbf{K}$ such that $x = y + t_1v_1 + ... + t_mv_m$. If $|x| \le R$ and $|y| \le R$, then $B(\mathbf{K}^m, 0, R) = B(\mathbf{K}^m, x, R) = B(\mathbf{K}^m, y, R)$ due to the ultrametric inequality. On the other hand, $B(\mathbf{K}^m, 0, R)$ is the additive group, hence $y - x \in B(\mathbf{K}^m, 0, R)$. Vectors v_j have coordinates $v_j = (v_j^1, ..., v_j^m)$, where $v_j^k \in \mathbf{K}$, consequently, $|v_j| = \max(|v_j^1|, ..., |v_j^m|)$. Thus, $|x-y| = \max(|t_1v_1^1 + ... + t_mv_m^1|, ..., |t_1v_1^m + ... + t_mv_m^m|) \le \max(|t_1v_1|, ..., |t_mv_m|)$. Choose $t_1, ..., t_m$ such that $|y + t_1v_1 + ... + t_kv_k| \le R$ also for each $k = 1, ..., m$. Therefore,

$|g_{v_j}(x) - g_{v_j}(y)| = |g_{v_j}(x) - g_{v_j}(y + t_1v_1 + ... + t_{m-1}v_{m-1}) + g_{v_j}(y + t_1v_1 + ... + t_{m-1}v_{m-1}) - ... - g_{v_j}(y + t_1v_1) + g_{v_j}(y + t_1v_1) - g_{v_j}(y)| \le \max_{k=1}^m |g_{v_j}(y + t_1v_1 + ... + t_kv_k) - g_{v_j}(y + t_1v_1 + ... + t_{k-1}v_{k-1})| \le \max(b, |t_1|^r, ..., |t_m|^r) \le \max(b, |x-y|^r)$ for each $|x| \le R$ and $|y| \le R$ as the consequence of Inequalities (2) and (4) and the ultrametric inequality for each $|x| \le R$ and $|y| \le R$, since $j = 1, ..., m$.

Further proceed by induction on n. From the preceding prove it follows, that the statement of this lemma is true for $n = m$. Put $\Omega = \Omega_0 \cup \{w\}$, where $w \notin \Omega_0$ and all elements of Ω are pairwise linearly independent over the field $\mathbf{K}$. Assume that the assertion of this lemma is true for Ω_0 and prove it for Ω. For $v \in \Omega_0$ denote $h_v(x, u) = h_v(x) = g_v(x + uw) - g_v(x)$, where $|x| \le R$, and $|x + uv| \le R$. For these values of x and $x + uv$ the function h_v satisfies Conditions $(1, 2)$. On the other hand, from (2) for $v = w$ and (3)

$|\sum_{v\in\Omega_0}(h_v(x)-h_v(y))| \le \max(|\sum_{v\in\Omega}(g_v(x+uw)-g_v(x))|, |\sum_{v\in\Omega}(g_v(y+uw)-g_v(y))|, |g_w((x+uw)-g_w(x)|, |g_w(y+uw)-g_w(y)|) \le \max(b,|u|^r)$
for each $|x|\le R$, $|y|\le R$ and $|uw|\le R$. Thus, $\{h_v : v\in\Omega_0\}$ satisfies Condition (3) with $\max(b,|u|^r)$ instead of b. By the induction hypothesis there exists $C_1 = const > 0$, which may depend only on Ω_0, r and R such that

(5) $|h_v(x)-h_v(y)| \le C_1\max(b,|u|^r,|x-y|^r)$

for each $|x|\le R$, $|y|\le R$ and $|uw|\le R$ and $v\in\Omega_0$. Take $y-x=(q-1)uw$ with $q\in\mathbf{K}$, $|q|>1$, hence $|q-1|>1$ and Inequality (5) will take the form:

(6) $|g_v(x+quw)-g_v(x+(q-1)uw)-g_v(x+uw)+g_v(x)| \le C_1\max(b,|(q-1)u|^r,|(q-1)uw|^r) \le C_2\max(b,|u|^r)$,

when $|x|\le R$, $|quw|\le R$ and $v\in\Omega_0$, where $C_2\ge C_1|q-1|^r\max(1,|w|^r)$. Now set $s(u):=g_v(x+uw)-g_v(x)$ for $v\in\Omega_0$ and from (2) and (6) and the ultrametric inequality it follows, that

$|s(qu)-qs(u)| \le C_2\max(b,|u|^r)$,

when $|quw|\le R$. In view of Lemma 19

(7) $|s(u)| = |g_v(x+uw)-g_v(x)| \le C_3\max(b,|u|^r)$

for each $|uw|\le R$, $|x|\le R$ and $v\in\Omega_0$, where $C_3=\max(a^{-r},|q|^{-1}a^rC_2|q|^{-r})$. Interchanging roles of w and one of $v\in\Omega_0$ we obtain (7) with w in place of v, that is, (4) is proved for each $v\in\Omega$.

33. Corollary. *Let $v_1,...,v_n$ be pairwise $\mathbf{K}$-linearly independent vectors in $\mathbf{K}^m$ and each subset consisting not less than m of these vectors has the $\mathbf{K}$-linear span coinciding with $\mathbf{K}^m$ and let g_k be locally bounded functions from $\mathbf{K}^m$ into $\mathbf{K}$, $0<r\le 1$. If $g_k\in Lip(v_k,r)$ for each k and $\sum_{k=1}^n g_k(x)=0$ identically by $x\in\mathbf{K}^m$, then $g_k\in Lip(r)$ for each k.*

Proof. If $c\in\mathbf{K}$, $c\ne 0$ is small enough, then the functions cg_k satisfy assumptions of Lemma 32 with $b=0$.

34. Remark. We can mention, that apart from the classical case over $\mathbf{R}$ this lemma is true also for $r=1$ due to the ultrametric inequality, which is stronger than the usual triangle inequality.

35. Definition. Let $v\in\mathbf{K}^b$ and $v\ne 0$. We say that a function $f:\mathbf{K}^b\to\mathbf{K}$ is continuous in the direction v if $f(x+tv)$ converges to $f(x)$ uniformly by x on bounded closed sets as t tends to zero.

Mention that in a particular case of a locally compact field $\mathbf{K}$ a bounded closed subset is compact.

36. Lemma. *Suppose that $f\in C^0(\mathbf{K}^b,\mathbf{K})$ and $\Upsilon^1 f(x,w,t)$ is continuous*

or uniformly continuous on $V^{[1]}$ *in the direction* $v^{[1]}$ *with* $v_2^{[1]} \ne 0$ *and* $v_3^{[1]} \ne 0$*, where* $V^{[1]} := \{(x,v,t) \in U^{[1]} : |v| = 1\}$*,* U *is open in* $\mathbf{K}^b$*. Then* $\Upsilon^1 f(x, v_2^{[1]}, t)$ *is continuous or uniformly continuous by* (x,t)*,* $(x,v,t) \in U^{[1]}$ *or* $(x,v,t) \in V^{[1]}$ *respectively.*

Proof. Assume the contrary, that $\Upsilon^1 f(x, v_2^{[1]}, t)$ is not continuous by (x,t). Making a shift in case of necessity we can suppose that $\Upsilon^1 f(x, v_2^{[1]}, t)$ is not continuous by (x,t) at 0 or is not uniformly continuous on $V^{[1]}$. Therefore, there exists a sequence $\{x_n^{[1]} \in (\mathbf{K}^b)^{[1]} : n \in \mathbf{N}\}$ such that $|\Upsilon^1 f(x_n^{[1]}) - \Upsilon^1 f(0)| > \epsilon$ for each n or with $x_0^{[1]} \in V^{[1]}$ instead of 0 and a family of sequences parametrized by $x_0^{[1]}$ and $\sup_{x_0^{[1]} \in V^{[1]}} |\Upsilon^1 f(x_n^{[1]}) - \Upsilon^1 f(x_0^{[1]})| > \epsilon$ correspondingly, where $\epsilon > 0$ is a constant, $x_n^{[1]} = (x_n, v_2^{[1]}, t_n)$, $\lim_{n\to\infty}(x_n, t_n) = x_0^{[1]}$. But in accordance with Definition 21 there exists $\delta > 0$ independent of n such that $|\Upsilon^1 f(x_n^{[1]} + v^{[1]}\tau) - \Upsilon^1 f(v^{[1]}\tau)| > \epsilon|\pi|$ or $\sup_{x_0^{[1]} \in V^{[1]}} |\Upsilon^1 f(x_n^{[1]} + v^{[1]}\tau) - \Upsilon^1 f(v^{[1]}\tau)| > \epsilon|\pi|$ for each n and each $\tau \in \mathbf{K}$ with $|\tau| \le \delta$. On the other hand, $\Upsilon^1 f(x_n + v_1^{[1]}\tau, w_n + v_2^{[1]}\tau, t_n + v_3^{[1]}\tau) - \Upsilon^1 f(v^{[1]}\tau) = [f(x_n + v_1^{[1]}\tau + (w_n + v_2^{[1]}\tau)(t_n + v_3^{[1]}\tau)) - f(x_n + v_1^{[1]}\tau)]/(t_n + v_3^{[1]}\tau) - [f(v_1^{[1]}\tau + v_2^{[1]}\tau v_3^{[1]}\tau) - f(v_1^{[1]}\tau)]/(v_3^{[1]}\tau)$, where $w_n = v_2^{[1]}$. But

$\lim_{n\to\infty}[f(x_n + v_1^{[1]}\tau + (w_n + v_2^{[1]}\tau)(t_n + v_3^{[1]}\tau))/(t_n + v_3^{[1]}\tau) - f(v_1^{[1]}\tau)/(v_3^{[1]}\tau)] = 0$ and

$\lim_{n\to\infty}[f(x_n + v_1^{[1]}\tau)/(t_n + v_3^{[1]}\tau) - f(v_1^{[1]}\tau)]/(v_3^{[1]}\tau) = 0$

for $v_3^{[1]}\tau \ne 0$ pointwise or uniformly respectively. If $v_3^{[1]}\tau = 0$, then $\Upsilon^1 f(x_n + v_1^{[1]}\tau, w_n + v_2^{[1]}\tau, t_n + v_3^{[1]}\tau) - \Upsilon^1 f(v^{[1]}\tau) = \Upsilon^1 f(x_n + v_1^{[1]}\tau, w_n + v_2^{[1]}\tau, 0) - \Upsilon^1 f(v_1^{[1]}\tau, w_n + v_2^{[1]}\tau, 0)$, but the latter difference tends to zero as τ tends to zero uniformly by n or also uniformly by the family of sequences parametrized by $x_0^{[1]}$ respectively in accordance with the supposition of lemma. Thus we get the contradiction with our supposition, hence $\Upsilon^1 f(x, v_2^{[1]}, t)$ is continuous or uniformly continuous by (x,t) correspondingly.

37. Definition. Denote by either $C^{[n]}_{\phi,b}(U,Y)$ or $C^n_{\phi,b}(U,Y)$ spaces of all functions $f \in C^{[n]}_\phi(U,Y)$ or $f \in C^n_\phi(U,Y)$ such that $f^{[k]}(x^{[k]})$ or $\bar\Phi^k f(x^{(k)})$ is uniformly continuous on a subset either $V^{[k]} := \{x^{[k]} \in U^{[k]} : |v_1^{[q]}| = 1; |\,_l v_2^{[q]} t_{q+1}| \le 1, |v_3^{[q]}| \le 1 \quad \forall l, q\}$ or $V^{(k)} := \{x^{(k)} \in U^{(k)} : |v_j| = 1 \quad \forall j\}$ for each $k = 0, 1, ..., n$ with finite norms either

$\|f\|_{[n]} := \|f\|_{[n],\phi} := \max(C, \sup_{k=0,\dots,n; x^{[k]} \in V^{[k]}} |f^{[k]}(x^{[k]})|)$ or

$\|f\|_n := \|f\|_{n,\phi} := \max(C, \sup_{k=0,\dots,n; x^{(k)} \in V^{(k)}} |\bar\Phi^k f(x^{(k)})|)$,

where $0 \le C < \infty$ is the least constant satisfying 29(1) or 29(2) for $\Upsilon^n f(x^{[n]})$ or $\bar{\Phi}^n f(x^{(n)})$ respectively instead of f. For $\phi(q) = q^r$ we denote $C^{[n]}_\phi(U,Y)$ by $C^{[n],r}(U,Y)$ and $C^n_{\phi,b}(U,Y)$ by $C^{n,r}_b(U,Y)$, $0 \le r \le 1$. For $r = 0$ we put $C^{[n],0} = C^{[n]}$, $C^{[n],0}_b = C^{[n]}_b$ and $C^{n,0} = C^n$, $C^{n,0}_b = C^n_b$, with $C = 0$ in the definition of the norm. As usually $C^{[\infty]}(U,Y) := \bigcap_{k=0}^\infty C^{[k]}(U,Y)$ and $C^\infty(U,Y) := \bigcap_{k=0}^\infty C^k(U,Y)$ and $C^{[\infty]}_b(U,Y) := \bigcap_{k=0}^\infty C^{[k]}_b(U,Y)$ and $C^\infty(U,Y) := \bigcap_{k=0}^\infty C^k_b(U,Y)$, where the topology of the latter two spaces is given by the family of the corresponding norms.

In the case of a locally compact field $\mathbf{K}$ and a compact clopen (closed and open at the same time) domain U we have $C^{[k]}_b(U,Y) = C^{[k]}(U,Y)$ and $C^k_b(U,Y) = C^k(U,Y)$, though for non locally compact $\mathbf{K}$ they are different $\mathbf{K}$-linear spaces.

38. Theorem. *Suppose that $f : \mathbf{K}^m \to \mathbf{K}$, $m \in \mathbf{N}$ and $f \circ u \in C^s_\phi(\mathbf{K},\mathbf{K})$ or $f \circ u \in C^s_{\phi,b}(\mathbf{K},\mathbf{K})$ or $f \circ u \in C^{[s]}_\phi(\mathbf{K},\mathbf{K})$ or $f \circ u \in C^{[s]}_{\phi,b}(\mathbf{K},\mathbf{K})$ for each $u \in C^\infty(\mathbf{K},\mathbf{K}^m)$ or $u \in C^\infty_b(\mathbf{K},\mathbf{K}^m)$ or $u \in C^{[\infty]}(\mathbf{K},\mathbf{K}^m)$ or $u \in C^{[\infty]}_b(\mathbf{K},\mathbf{K}^m)$, where s is a nonnegative integer, $\phi : (0,\infty) \to (0,\infty)$ such that $\lim_{y\to 0} \phi(y) = 0$, then $f \in C^s(\mathbf{K}^m,\mathbf{K})$ or $f \in C^s_b(\mathbf{K}^m,\mathbf{K})$ or $f \in C^{[s]}(\mathbf{K}^m,\mathbf{K})$ or $f \in C^{[s]}_b(\mathbf{K}^m,\mathbf{K})$ respectively.*

Proof. In view of Lemma 21 it is sufficient to prove that $\bar{\Phi}^n f(x; e_{j(1)},...,e_{j(n)}; t_1,...,t_n)$ is in $C^0(U^{(n)}_{j(1),...,j(n)},Y)$ or $C^0_b(V^{(n)}_{j(1),...,j(n)},Y)$ or $\Upsilon^n f(x^{[n]})$ is in $C^0(U^{[n]}_{j(0),...,j(n)},Y)$ or $C^0_b(V^{[n]}_{j(0),...,j(n)},Y)$ respectively for each $n = 1,2,...,s$ and each $j(1),...,j(n) \in \{1,...,m\}$ or $j(i) \in \{1,...,m(i)\}$, $i = 0,1,...,n$. If $\Upsilon^{m+1} f$ or $\bar{\Phi}^{m+1} f$ is locally bounded, then $\Upsilon^m f$ or $\bar{\Phi}^m f$ is continuous respectively. Applying Lemma 27 by induction we get that $\Upsilon^n f(x^{[n]})$ is a locally bounded function on $(\mathbf{K}^m)^{[n]}$ and $\bar{\Phi}^n f(x^{(n)})$ is a locally bounded function on $(\mathbf{K}^m)^{(n)}$. In view of Lemma 30 each $\Upsilon^k f(x^{[k]})$ or $\bar{\Phi}^k f(x^{(k)})$ is continuous in each direction v for each $k = 1,...,s$, where $v \in (\mathbf{K}^m)^{[k]}$ or $v \in (\mathbf{K}^m)^{(k)}$ correspondingly. On the other hand, by induction on k we have that in accordance with Lemma 36 $\Upsilon^k f(x^{[k]})$ or $\bar{\Phi}^k f(x; e_{j(1)},...,e_{j(k)}; t_1,...,t_k)$ is continuous on $U^{[k]}_{j(0),...,j(k)}$ or $U^{(k)}_{j(1),...,j(k)}$ or bounded uniformly continuous respectively on $V^{[k]}_{j(0),...,j(k)}$ or $V^{(k)}_{j(1),...,j(k)}$ for bounded U for each $j(1),...,j(k) \in \{1,...,m\}$ or $j(i) \in \{1,...,m(i)\}$ for all $i = 0,1,...,k$.

39. Theorem. *Let $f : \mathbf{K}^m \to \mathbf{K}^n$, $m,n \in \mathbf{N}$. Let also $f \circ u \in C^{s,r}(\mathbf{K},\mathbf{K}^n)$ or $f \circ u \in C^{s,r}_b(\mathbf{K},\mathbf{K}^n)$ or $C^{[s],r}(\mathbf{K},\mathbf{K}^n)$ or $C^{[s],r}_b(\mathbf{K},\mathbf{K}^n)$ for*

each $u \in C^\infty(\mathbf{K},\mathbf{K}^m)$ or $u \in C^\infty_b(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K},\mathbf{K}^m)$ correspondingly, where s is a nonnegative integer, $0 \le r \le 1$, then $f \in C^{s,r}(\mathbf{K}^m,\mathbf{K}^n)$ or $f \in C^{s,r}_b(\mathbf{K}^m,\mathbf{K}^n)$ or $C^{[s],r}(\mathbf{K}^m,\mathbf{K}^n)$ or $C^{[s],r}_b(\mathbf{K}^m,\mathbf{K}^n)$ respectively.

Proof. If $s=0$ and $0 \le r \le 1$, then the assertion of this theorem follows from Lemmas 27 and 30. For $r>0$ by Theorem 38 $f \in C^s(\mathbf{K}^m,\mathbf{K}^n)$ or $f \in C^s_b(\mathbf{K}^m,\mathbf{K}^n)$ or $C^{[s]}(\mathbf{K}^m,\mathbf{K}^n)$ or $C^{[s]}_b(\mathbf{K}^m,\mathbf{K}^n)$ respectively. From Lemma 21 we infer, that it is sufficient to prove that $\bar{\Phi}^n f(x;e_{j(1)},...,e_{j(n)};t_1,...,t_n)$ is in $C^{0,r}(U^{(n)}_{j(1),...,j(n)},Y)$ or $C^{0,r}_b(V^{(n)}_{j(1),...,j(n)},Y)$ or $\Upsilon^n f(x^{[n]}) \in C^{0,r}(U^{[n]}_{j(0),...,j(n)},Y)$ or $C^{0,r}_b(V^{[n]}_{j(0),...,j(n)},Y)$ respectively for each $n=1,2,...,s$ and each $j(1),...,j(n) \in \{1,...,m\}$, $j(i) \in \{1,...,m(i)\}$, $i=0,1,...,n$. Prove this by induction by n. For $n=0$ it was proved above. Let it be true for $n=0,...,k$ and prove it for $n=k+1 \le s$. For this consider Formula 10(1) or 9(1). On the right hand side of it all terms having a total degree of f by operators B or A less than $k+1$ are in $C^{0,r}(U^{(n)},Y)$ or $C^{0,r}_b(V^{(n)},Y)$ or $C^{0,r}(U^{[n]},Y)$ or $C^{0,r}_b(V^{(n)},Y)$ respectively by the induction hypothesis, since $u \in C^\infty(\mathbf{K},\mathbf{K}^m)$ or $u \in C^\infty_b(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K},\mathbf{K}^m)$ correspondingly. Therefore, it remains to prove, that the sum

(i) $[\sum_{j_1,...,j_n}(B_{j_n,v^{(n-1)},t_n}...B_{j_1,v^{(0)},t_1}f \circ u)(\bar{\Phi}^1 \circ p_{j_n}\hat{S}_{j_{n-1}+1,v^{(n-2)}t_{n-1}} ...\hat{S}_{j_1+1,v^{(0)}t_1}u^{n-1})(P_n\bar{\Phi}^1 \circ p_{j_{n-1}}\hat{S}_{j_{n-2}+1,v^{(n-3)},t_{n-2}}...\hat{S}_{j_1+1,v^{(0)}t_1}u^{n-2})...(P_n...P_2\bar{\Phi}^1 \circ p_{j_1}u)]$

is in $C^{0,r}(U^{(n)},Y)$ or $C^{0,r}_b(V^{(n)},Y)$ or corresponding sum by compositions of $A_{j_k,v^{[k-1]},t_k}$ is in $C^{0,r}(U^{[n]},Y)$ or $C^{0,r}_b(V^{[n]},Y)$ respectively. In accordance with the proof above it is sufficient to demonstrate this for $v^{(n-1)}=(e_{j(1)},...,e_{j(n)})$ for each $j(1),...,j(n) \in \{1,...,m\}$ or $v^{[i]}=e_{j(i)}$ with $j(i) \in \{1,...,m(i)\}$ and $i=0,1,...,n$, where $v^{(l)}_0 = v_{l+1} = e_{j(l+1)}$, $l=0,...,n-1$. By the induction hypothesis $\bar{\Phi}^l f(x;e_{j(1)},...,e_{j(l)};t_1,...,t_l)$ is in $C^{0,r}(U^{(l)}_{j(1),...,j(l)},Y)$ or $C^{0,r}_b(V^{(l)}_{j(1),...,j(l)},Y)$ or $\Upsilon^l f(x^{[l]}) \in C^{0,r}(U^{[l]}_{j(0),...,j(l)},Y)$ or $C^{0,r}_b(V^{[l]}_{j(0),...,j(l)},Y)$ respectively for each $l=1,2,...,k$ and each $j(1),...,j(l) \in \{1,...,m\}$, $j(i) \in \{1,...,m(i)\}$, $i=0,...,n$. In view of Corollary 18 and Lemma 32 functions $\bar{\Phi}^n f(x;e_{j(1)},...,e_{j(n)};t_1,...,t_n)$ or $\Upsilon^n f(x^{[n]})$ belong to $Lip(v,r)$ by $(x,t_1,...,t_n)$ or $x^{[n]} \in U^{[n]}_{j(0),...,j(n)}$, where $v=(e_{j(n)};l_k) \in \mathbf{K}^{m+n}$, $e_j \in \mathbf{K}^m$, $l_k=(0,...,0,1,0,...,0) \in \mathbf{K}^n$ with 1 on the k-th place, or $v=(e_{j(0)},...,e_{j(n)};l_k)$ with $e_{j(i)} \in \mathbf{K}^{m(i)}$ respectively. By Corollary 18 each $\bar{\Phi}^n f(x;e_{j(1)},...,e_{j(n)};t_1,...,t_n)$ or $\Upsilon^n f(x^{[n]})|_{U^{[n]}_{j(0),...,j(n)}}$

belongs to $Lip(r)$ by $(x, t_1, ..., t_n)$ or in addition bounded uniformly lipschitzian on $V^{(n)}_{j(1),...,j(n)}$ or $V^{[n]}_{j(0),...,j(n)}$ respectively. In accordance with Lemma 21 this proves the theorem.

40. Theorem. *Let $f : \mathbf{K}^m \to \mathbf{K}^l$, $m \in \mathbf{N}$. Suppose also that $f \circ u \in C^\infty(\mathbf{K}, \mathbf{K})$ or $f \circ u \in C^\infty_b(\mathbf{K}, \mathbf{K})$ or $C^{[\infty]}(\mathbf{K}, \mathbf{K})$ or $C^{[\infty]}_b(\mathbf{K}, \mathbf{K})$ for each $u \in C^\infty(\mathbf{K}, \mathbf{K}^m)$ or $u \in C^\infty_b(\mathbf{K}, \mathbf{K}^m)$ or $C^{[\infty]}(\mathbf{K}, \mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K}, \mathbf{K}^m)$, then $f \in C^\infty(\mathbf{K}^m, \mathbf{K}^l)$ or $f \in C^\infty_b(\mathbf{K}^m, \mathbf{K}^l)$ or $C^{[\infty]}(\mathbf{K}^m, \mathbf{K}^l)$ or $C^{[\infty]}_b(\mathbf{K}^m, \mathbf{K}^l)$ respectively.*

Proof. Apply either Theorem 39 for each $s \in \mathbf{N}$ and $r = 0$ or Theorem 38 for each $s \in \mathbf{N}$ and $\phi(q) = q^r$ with $0 < r < 1$, since $C^{s,r}(U, Y) \subset C^{s+1}(U, Y)$ and $C^{s,r}_b(U, Y) \subset C^{s+1}_b(U, Y)$ and $C^\infty(U, Y) := \bigcap_{n=0}^\infty C^n(U, Y)$ and $C^\infty_b(U, Y) := \bigcap_{n=0}^\infty C^n_b(U, Y)$ and $C^{[\infty]}(U, Y) := \bigcap_{n=0}^\infty C^{[n]}(U, Y)$ and $C^{[\infty]}_b(U, Y) := \bigcap_{n=0}^\infty C^{[n]}_b(U, Y)$.

41. Theorem. *Let $h_j(y)$ be $C^\infty(\mathbf{K}, \mathbf{K})$ functions such that*

(1) *$h_j(0) = 0$ for each $j = 0, 1, ..., m$,*

(2) *$\lim_{0 \ne y \to 0} h_{j-1}(y)/h_j(y)^n = 0$ for each $n \in \mathbf{N}$ and $j = 1, ..., m$,*

(3) *$\lim_{0 \ne y \to 0} h_m(y)/y^n = 0$ for every $n \in \mathbf{N}$. Put $h(y) = (h_1(y), ..., h_m(y))$ and suppose that $g \in C^\infty(\mathbf{K}^b, \mathbf{K})$ is not identically zero and $g(x) = 0$ for each $|x| > 1$. Define $f : \mathbf{K}^{m+1} \to \mathbf{K}$ by the formula:*

(4) *$f(x, y) = g((x - h(y))/h_0(y))$ for each $y \ne 0$ and $x \in \mathbf{K}^m$, $f(x, 0) = 0$ for each x. Then $f \circ u \in C^\infty(\mathbf{K}^m, \mathbf{K})$ for each locally analytic function $u : \mathbf{K}^m \to \mathbf{K}^{m+1}$, but f is discontinuous.*

Proof. First demonstrate that f is not continuous at $(0, 0)$. We have $f(x, 0) = 0$. But take a sequence (x_n, y_n) such that $\lim_{n\to\infty}(x_n, y_n) = 0$ with $\epsilon \le |(x_n - h(y_n))/h_0(y_n)| \le 1$ and $|g(z_n)| \ge \delta$ with $z_n := (x_n - h(y_n))/h_0(y_n)$, which is possible since $\lim_{0 \ne y \to 0} |h(y)/h_0(y)| = \infty$ and g is continuous and non zero, where $\epsilon > 0$ and $\delta > 0$ are constants. For this sequence we have $|f(x_n, y_n)| \ge \delta$ for each n. But for the sequence (x_n, y_n) such that $|(x_n - h(y_n))/h_0(y_n)| > 1$ we have $f(x_n, y_n) = 0$, since $g(z_n) = 0$ for $|z_n| > 1$. Thus f is discontinuous at $(0, 0)$.

Now take a locally analytic function $u : \mathbf{K}^m \to \mathbf{K}^{m+1}$ and consider the composition $f \circ u$. Take a nontrivial analytic function $w(x, y)$ from a neighborhood of zero in $\mathbf{K}^{m+1}$ into $\mathbf{K}$ such that $w \circ u(y) = 0$ in a neighborhood of $y_0 \in \mathbf{K}^m$, where $u(y_0) = 0$. Prove that for functions $h_j(y)$ satisfying Conditions $(1-3)$ there exist constants $C > 0$ and $\delta > 0$ such that

(5) $|w(x, y)| \ge C|h_0(y)|^n$,

when $|x - h(y)| \le |h_0(y)|$, $0 < |y| < \delta$. If prove (5), then from $(x, y) = u(t)$ for t in a neighborhood of y_0 it follows, that $w(x, y) = 0$ and by (5) we have that $|x - h(y)| > |h_0(y)|$, hence $f(x, y) = 0$.

Consider an analytic function q from a neighborhood of zero in $\mathbf{K}^l$ into $\mathbf{K}$. Then we can write it in the form:

(6) $q(x) = x_1^k s(x_2, ..., x_l) + x_1^{k+1} r(x)$,

where s and r are analytic functions and s is not identically zero, $1 \le l \le m$, $x_1, ..., x_l \in \mathbf{K}$. Then $|q(h(y))| \ge |h_1(y)|^k (|s(h_2(y), ..., h_l(y))| - C|h_1(y)|) \ge |h_1(y)|^k (C|h_2(y)|^n - C|h_1(y)|) \ge C|\pi h_1(y)|^{k+n}$

for each y with $0 < |y| < \delta$ with suitable $\delta > 0$ and $n \in \mathbf{N}$, where $\pi \in \mathbf{K}$, $0 < |\pi| < 1$. Then induction by l gives from (6) that for an arbitrary nontrivial analytic function q from a neighborhood of zero in $\mathbf{K}^m$ into $\mathbf{K}$ there exist constants $C > 0$ and $\delta > 0$ and $n \in \mathbf{N}$ such that

(7) $|q(h(y))| \ge C|h_1(y)|^n$ for each $0 < |y| < \delta$.

In particular, from (7) it follows, that there exist $C > 0$, $\delta > 0$ and $n \in \mathbf{N}$ such that

(8) $|w(h(y), y)| \ge C|h_1(y)|^n$ for each $0 < |y| < \delta$.

It remains to show that (8) implies (5). Take $C > 0$ so large that $|grad_x w(x, y)| \le C$ in some neighborhood of zero and assume that $|x - h(y)| \le |h_0(y)|$. Then there exists $\delta > 0$ such that $|w(x, y)| \ge |w(h(y), y)| - C|x - h(y)| \ge C|h_1(y)|^n - C|h_0(y)| \ge C|\pi h_1(y)|^n$ for each $0 < |y| < \delta$. Thus $f \circ u = 0$ in a neighborhood of each point $y_0 \in \mathbf{K}^m$ such that $u(y_0) = 0$.

For example, we can take either $h_j(y) = \sum_n a_n \pi^{n^2(m-j+1)+n}$ for each $y = \sum_n a_n \pi^n \in \mathbf{K}$, where $a_n \in \mathbf{K}$ belong to the finite set of representatives of distinct classes in the finite factor field $B(\mathbf{K}, 0, 1)/B(\mathbf{K}, 0, |\pi|)$, $\pi \in \mathbf{K}$, $0 < |\pi| < 1$, $\mathbf{K}$ is a locally compact field of zero characteristic and $|\pi|$ is the largest generator among those less than one of the valuation group $\Gamma_{\mathbf{K}}$ of $\mathbf{K}$, or $h_j(y) = \sum_n a_n \theta^{n^2(m-j+1)+n}$ for each $y = \sum_n a_n \theta^n \in \mathbf{F}_{p^k}(\theta)$, where $a_n \in \mathbf{F}_{p^k}$, p is a prime number, $k \in \mathbf{N}$, $\mathbf{F}_{p^k}(\theta)$ is a locally compact field of characteristic $char(\mathbf{F}_{p^k}(\theta)) = p > 0$, $\mathbf{F}_{p^k}$ is a finite field of p^k elements.

42. Theorem. *There exists a discontinuous function $f : \mathbf{K}^m \to \mathbf{K}$ such that $f \circ u \in C^\infty(\mathbf{K}^{m-1}, \mathbf{K})$ for each locally analytic function $u : \mathbf{K}^{m-1} \to \mathbf{K}^b$, where $m \ge 2$.*

Proof. This theorem follows from Theorem 41. Another its proof is the following. Let $f \in C^\infty(\mathbf{K}^2 \setminus \{0\}, \mathbf{K})$ and let f be non constant with $f(x_1, x_2) = 0$, when $x_1 x_2 = 0$. For simplicity let $\mathbf{K}$ be a locally compact field of zero characteristic. Take an analytic function $g : \mathbf{K} \to \mathbf{K}$ such that

$\lim_{|y|\to\infty} g(y)=0$. Such functions exist due to Example 43.1 of Section 43 in [19]. Moreover, they can be chosen such that $|g(y)|\le \epsilon_j$ for $|y|=|\pi|^{-j}$ for each $j=0,1,2,...$ and a sequence $\{\epsilon_j>0 : j\}$, which in particular may also tend to zero. Then consider the function $g(1/x_2)$ and put $h(x_1,x_2):= f(x_1,g(1/x_2))$, where f is homogeneous of degree zero. Since $f\in C^\infty(\mathbf{K}^2\setminus\{0\},\mathbf{K})$ it remains to show that $f\circ u\in C^\infty$ in a neighborhood of $y=0$, if $u(0)=(0,0)$. If u_1 coincides with zero, then h is identically zero. If $u_1(0)=0$ and u_1 is not identically zero, then due to analyticity there exists $k\in\{1,2,...\}$ such that $u_1(t)=t^k v_1(t)$ and v_1 is locally analytic and $v_1(0)\ne 0$. From $u_2(0)=0$ it follows that $g(1/u_2(t))=t^k v_2(t)$, where v_2 is locally analytic and $v_2(0)=0$. We can take, for example, $\epsilon_j=|\pi|^{j^2}$. Since $f(x_1,0)=0$ and f is homogeneous of degree zero, then $h(u_1(t),u_2(t))=f(t^k v_1(t),t^k v_2(t))= f(v_1(t),v_2(t))$ for each $t\in\mathbf{K}$. Since $v_1(0)\ne 0$, then $f\circ u\in C^\infty$ in a neighborhood of zero.

43. Remark. In the non archimedean case analogs of classical theorems over $\mathbf{R}$ such as 3 and 10 [2] are not true due to the ultrametric inequality for the non archimedean norm, and since if a function f is homogeneous, then $\bar\Phi^k$ need not be homogeneous for $k\ge 1$. Theorem 2 from [2] in the non archimedean case is true in the stronger form due to the ultrametric inequality (see Theorem 38 above). The notion of quasi analyticity used in the classical case in [2] has not sense in the non archimedean case because of the necessity to operate with $\bar\Phi^k f$ instead of $D^k f$. It leads naturally to the local analyticity in the non archimedean case. In the latter case the exponential function has finite radius of convergence on $\mathbf{K}$ with $char(\mathbf{K})=0$. Therefore, in the proof of Theorem 40 it was used specific feature of the non archimedean analysis of analytic functions for which an analog of the Louiville theorem is not true (see also [19]).

Using the particular variant of Theorem 38 with $s=r=0$ it is easy to prove the following theorem.

Theorem. *Let $f:\mathbf{K}^m\to\mathbf{K}^l$, $f\circ u\in C^n(\mathbf{K}^2,\mathbf{K}^l)$ or $f\circ u\in C^n_b(\mathbf{K}^2,\mathbf{K}^l)$ or $C^{[n]}(\mathbf{K}^2,\mathbf{K}^l)$ or $C^{[n]}_b(\mathbf{K}^2,\mathbf{K}^l)$ for each $u\in C^\infty(\mathbf{K}^2,\mathbf{K}^m)$ or $u\in C^\infty_b(\mathbf{K}^2,\mathbf{K}^m)$ or $C^{[\infty]}(\mathbf{K}^2,\mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K}^2,\mathbf{K}^m)$, where $m\ge 2$ and $n\ge 1$. Then $f\in C^n(\mathbf{K}^m,\mathbf{K}^l)$ or $f\in C^n_b(\mathbf{K}^m,\mathbf{K}^l)$ or $C^{[n]}(\mathbf{K}^m,\mathbf{K}^l)$ or $C^{[n]}_b(\mathbf{K}^m,\mathbf{K}^l)$ correspondingly.*

Proof. Put $u(y)=\sum_{j=1}^m y_1^j e_j + w(y_2)$, where $y=(y_1,y_2)\in\mathbf{K}^2$, $e_j\in\mathbf{K}^m$, $w\in C^\infty(\mathbf{K},\mathbf{K}^m)$ or $C^\infty_b(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}(\mathbf{K},\mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K},\mathbf{K}^m)$. Therefore,

$u \in C^\infty(\mathbf{K}^2, \mathbf{K}^m)$ or $C^\infty_b(\mathbf{K}^2, \mathbf{K}^m)$ or $C^{[\infty]}(\mathbf{K}^2, \mathbf{K}^m)$ or $C^{[\infty]}_b(\mathbf{K}^2, \mathbf{K}^m)$. In view of Formula 10(1) or 9(1) and Lemmas 11, 12 or Corollary 14 for $\bar{\Phi}^k f \circ u(y^{(k)})$ or $\Upsilon^n f \circ u$ by induction we get that each $\bar{\Phi}^k f(w(y_2), e_{j(1)}, ..., e_{j(k)}; t_1, ..., t_k)$ or $\Upsilon^n f(x^{[n]})|_{V^{[n]}_{j(0),...,j(n)}}$ with $x = w(y_2)$ is continuous or uniformly continuous. Therefore, from Theorem 39 with $s = r = 0$ it follows, that each $\bar{\Phi}^k f(x; e_{j(1)}, ..., e_{j(k)}; t_1, ..., t_k)$ or $\Upsilon^n f(x^{[k]})$ is continuous on $U^{(k)}_{j(1),...,j(k)}$ or $U^{[n]}_{j(0),...,j(n)}$ or uniformly continuous on $V^{(k)}_{j(1),...,j(k)}$ or $V^{[k]}_{j(0),...,j(k)}$ respectively for each $k = 1, ..., n$, hence by Lemma 21 $f \in C^n(\mathbf{K}^m, \mathbf{K}^l)$ or $f \in C^n_b(\mathbf{K}^m, \mathbf{K}^l)$ or $C^{[n]}(\mathbf{K}^m, \mathbf{K}^l)$ or $C^{[n]}_b(\mathbf{K}^m, \mathbf{K}^l)$ correspondingly.

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